Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Functional Analysis 236 (2006) 299–350

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Toeplitz operators and Hamiltonian torus actions ☆

L. Charles

Université Pierre et Marie Curie – Paris 6, UMR 7586 IMJ, F-75005 Paris, France

Received 23 April 2005; accepted 10 October 2005

Available online 17 February 2006

Communicated by Richard B. Melrose

Abstract

This paper is devoted to semi-classical aspects of symplectic reduction. Consider a compact prequantizable Kähler manifold M with a Hamiltonian torus action. In the seminal paper [V. Guillemin, S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math. 67 (3) (1982) 515–538], Guillemin and Sternberg introduced an isomorphism between the invariant part of the quantum space associated to M and the quantum space associated to the symplectic quotient of M, provided this quotient is non-singular. We prove that this isomorphism is a Fourier integral operator and that the Toeplitz operators of M descend to Toeplitz operators of the reduced phase space. We also extend these results to the case where the symplectic quotient is an orbifold and estimate the spectral density of a reduced Toeplitz operator, a result related to the Riemann–Roch–Kawasaki theorem.

Keywords: Symplectic reduction; Quantization; Toeplitz operator; Orbifold; Spectral density

1. Introduction

Consider a symplectic manifold (M, ω) with a Hamiltonian action of a d-dimensional torus $\mathbb{T}^d$. Let μ be a momentum map. Following Marsden, Weinstein [18], if λ is a regular value of μ, the reduced space

$$M_r := \mu^{-1}(\lambda)/\mathbb{T}^d$$

☆ This work has been partially supported by the European Commission under the Research Training Network (Mathematical Aspects of Quantum Chaos) No. HPRN-CT-2000-00103 of the IHP Programme.

E-mail address: charles@math.jussieu.fr.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2005.10.011

is naturally endowed with a symplectic form ω_r. The quantum analogue of this reduction has been the subject of important studies, starting from the paper [13] of Guillemin and Sternberg, and has led to many versions of the "quantization commutes with reduction" theorem. In most of these articles, the quantization is defined as a Riemann–Roch number or the index of a spin-c Dirac operator which represents the dimension of a virtual quantum space, cf. the review article [23]. The relationships between deformation quantization and symplectic reduction have also been considered [11,26].

This paper is devoted to the quantum aspects of symplectic reduction in the semi-classical setting. Here the quantization consists of a Hilbert space with a semi-classical algebra of operators. More precisely, we assume that M is compact, Kähler and endowed with a prequantization bundle $L \to M$, i.e. a Hermitian line bundle with a connection of curvature $-i\omega$. For every positive integer k, let us define the quantum space $\mathcal{H}_k$ as the space of holomorphic sections of L^k. The semi-classical limit is $k \to \infty$ and the operators we will consider are the Toeplitz operators, introduced by Berezin in [2]. The application of microlocal techniques in this context started with Boutet de Monvel and Guillemin [4]. This point of view made it possible to extend many results known for the pseudodifferential operators with small parameter to the Toeplitz operators, as for instance, the trace formula [3] and the Bohr–Sommerfeld conditions [7].

Assume that the torus action preserves the complex structure of M. Then following Kostant and Souriau we can associate to the components $(\mu_1, \dots, \mu_d)$ of the moment map μ some commuting operators $\mathrm{M}_1, \dots, \mathrm{M}_d : \mathcal{H}_k \to \mathcal{H}_k$. Suppose that $\lambda = (\lambda_1, \dots, \lambda_d)$ is a joint eigenvalue of these operators when $k = 1$ and that the torus action on $\mu^{-1}(\lambda)$ is free. The following theorem is a slight reformulation of the main result of Guillemin and Sternberg.

Theorem 1.1. (See [13]) *M_r inherits a natural Kähler structure and a prequantization bundle L_r, which defines quantum spaces $\mathcal{H}_{r,k}$. Furthermore, for any k, there exists a natural vector space isomorphism V_k from*

$$\mathcal{H}_{\lambda,k} := \bigcap_{i=1}^{d} \mathrm{Ker}(\mathrm{M}_i - \lambda_i)$$

onto $\mathcal{H}_{r,k}$.

The various quantum spaces have natural scalar products induced by the Hermitian structure of the prequantization bundles and the Liouville measures, but unfortunately the isomorphism V_k is not necessarily unitary. So we will use

$$\mathrm{U}_k := \mathrm{V}_k \big(\mathrm{V}_k^* \mathrm{V}_k\big)^{-1/2} : \mathcal{H}_{\lambda,k} \to \mathcal{H}_{r,k}$$

instead. Our first result relates the Toeplitz operators of M with the Toeplitz operators of M_r.

Theorem 1.2. *Let $(\mathrm{T}_k : \mathcal{H}_k \to \mathcal{H}_k)_{k \in \mathbb{N}^*}$ be a Toeplitz operator of M which commutes with $\mathrm{M}_1, \dots, \mathrm{M}_d$, and with principal symbol $f \in C^\infty(M)$. Then*

$$\big(\mathrm{U}_k \mathrm{T}_k \mathrm{U}_k^* : \mathcal{H}_{r,k} \to \mathcal{H}_{r,k}\big)_k$$

is a Toeplitz operator of M_r. Furthermore, f is $\mathbb{T}^d$-invariant and the principal symbol f_r of $(\mathrm{U}_k\mathrm{T}_k\mathrm{U}_k^*)$ is such that $p^* f_r = j^* f$, where p and j are respectively the projection $\mu^{-1}(\lambda) \to M_r$ and the embedding $\mu^{-1}(\lambda) \to M$.

When the torus action on $\mu^{-1}(\lambda)$ is not free but locally free, the reduced space M_r is not a manifold, but an orbifold. These spaces with finite quotient singularities were first introduced by Satake in [22]. Many results or notions of differential geometry have been generalized to orbifolds: index theorem [17], fundamental group [24], string theory [21]. Not surprisingly, Theorems 1.1 and 1.2 are still valid in this case. Motivated by this, we prove the basic properties of the Toeplitz operators on the orbifold M_r: description of their Schwartz kernel and the symbolic calculus. Our second main result is the estimate of the spectral density of a Toeplitz operator on the orbifold M_r. This simple result in the manifold case involves here oscillatory contribution of the inertia orbifolds or twisted sectors associated to M_r and is related to the Kawasaki–Riemann–Roch theorem [17], cf. Theorems 2.3 and 6.9 for precise statements.

With a view towards application, we also consider the simple case where M is $\mathbb{C}^n$ with a linear circle action whose momentum map is a proper harmonic oscillator. The quantum data associated with $\mathbb{C}^n$ are defined by the Bargmann representation and the reduced space is a twisted projective space. Actually, this is nearly a particular case of the previous setting, except that $\mathbb{C}^n$ is not compact. As a corollary of Theorem 1.2, the spectral analysis of an operator commuting with the quantum harmonic oscillator is reduced to that of a Toeplitz operator on a projective space. In collaboration with San Vu Ngoc, we plan to apply this to the semi-excited spectrum of a Schrödinger operator with a non-degenerate potential well.

We also prove that the isomorphism V_k of Theorem 1.1 and its unitarization U_k are Fourier integral operators. Thus we can interpret Theorem 1.2 as a composition of Fourier integral operators with underlying compositions of canonical relations. Actually our proof of Theorem 1.2 is elementary in the sense that it relies on the geometric properties of the isomorphism V_k and does not use the usual tools of microlocal analysis. But with the more general point of view of Fourier integral operators, we hope that we can extend the "quantization commutes with reduction" theorems by using microlocal techniques. For instance, Theorems 1.1 and 1.2 should hold with general Toeplitz operators M_i whose joint principal symbol define a momentum map without assuming that the action preserves the complex structure. Also the Kähler structure is certainly not necessary. This microlocal approach is also related to another paper of Guillemin and Sternberg [14] (cf. Sections 4.3 and 5.3 for a comparison with our results).

The organization of the paper is as follows. Section 2 contains detailed statements of our main results for the harmonic oscillator on $\mathbb{C}^n$. In Section 3, we introduce our set-up in the compact Kähler case and recall the results of [13] proving that the reduced quantum space is isomorphic to the joint eigenspace. Section 4 contains the statements and proofs of our main results for the reduction of Toeplitz operators. In Section 5, we interpret these results as compositions of Fourier integral operators. Sections 6 and 7 are devoted to the Toeplitz operators on Kähler orbifolds.

2. Statement of the results for the harmonic oscillator

Assume $\mathbb{C}^n$ is endowed with the usual symplectic 2-form $\omega = i(dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n)$. Let H be the harmonic oscillator

$$H := \mathfrak{p}_1|z_1|^2 + \cdots + \mathfrak{p}_n|z_n|^2,$$

where $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are positive relatively prime integers. Consider the scalar product

$$(\Psi, \Psi')_{\mathbb{C}^n} = \int_{\mathbb{C}^n} e^{-\hbar^{-1}|z|^2} \Psi(z).\bar{\Psi}'(z) \frac{|dz.d\bar{z}|}{n!}, \tag{1}$$

where Ψ, Ψ' are functions on $\mathbb{C}^n$ and $|z|^2 = |z_1|^2 + \dots + |z_n|^2$. The Bargmann space $\mathcal{H}$ is the Hilbert space of holomorphic functions Ψ on $\mathbb{C}^n$ such that $(\Psi, \Psi)_{\mathbb{C}^n} < \infty$. The quantum harmonic oscillator is the unbounded operator of $\mathcal{H}$

$$\mathrm{H} := \hbar(\mathfrak{p}_1 z_1 \partial_{z_1} + \dots + \mathfrak{p}_n z_n \partial_{z_n}) \tag{2}$$

with domain the space of polynomials on $\mathbb{C}^n$.

2.1. Symplectic reduction

The Hamiltonian flow of H induces an action of S^1 on the level set $P := \{H = 1\}$

$$S^1 \times P \to P, \quad \theta, z \to l_\theta.z = \left(z_1 e^{i\theta\mathfrak{p}_1}, \dots, z_n e^{i\theta\mathfrak{p}_n}\right) \text{ if } z = (z_1, \dots, z_n).$$

Define the reduced space M_r as the quotient P/S^1. If $\mathfrak{p}_1 = \dots = \mathfrak{p}_n = 1$, the action is free, M_r is a manifold and the projection $P \to M_r$ is the Hopf fibration. When the $\mathfrak{p}_i$ are not all equal to 1, the action is not free, but locally free. Hence M_r is not a manifold, but an orbifold. In any cases, M_r is naturally endowed with a symplectic 2-form ω_r.

We may also define a complex structure on the space M_r by viewing it as a complex quotient. Consider the holomorphic action of $\mathbb{C}^*$

$$\mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n, \qquad u, (z_1, \dots, z_n) \to \left(z_1 u^{\mathfrak{p}_1}, \dots, z_n u^{\mathfrak{p}_n}\right). \tag{3}$$

Each $\mathbb{C}^*$-orbit of $\mathbb{C}^n - \{0\}$ intersects P in a S^1-orbit, which identifies M_r with $\mathbb{C}^n - \{0\}/\mathbb{C}^*$. This quotient is called a twisted projective space, the standard projective space is obtained when $\mathfrak{p}_1 = \dots = \mathfrak{p}_n = 1$. The complex structure is compatible with the symplectic form ω_r. So M_r is a Kähler orbifold.

2.2. Quantum reduction

H has a discrete spectrum given by

$$\mathrm{Sp}(\mathrm{H}) = \left\{\hbar\left(\mathfrak{p}_1\alpha(1) + \dots + \mathfrak{p}_n\alpha(n)\right);\ \alpha \in \mathbb{N}^n\right\}.$$

Hence 1 is an eigenvalue only if $\hbar$ is of the form $1/k$ with k a positive integer. Since we will only consider the eigenvalue 1, we assume from now on that

$$\hbar = 1/k \quad \text{with } k \in \mathbb{N}^*$$

and use the large parameter k instead of the small parameter $\hbar$. We denote by $\mathcal{H}_k$ and H_k the Bargmann space and the quantum harmonic oscillator. The vector space

$$\mathcal{H}_{1,k} := \mathrm{Ker}(\mathrm{H}_k - 1)$$

is generated by the monomials z^α such that $\mathfrak{p}_1\alpha(1)+\cdots+\mathfrak{p}_n\alpha(n)=k$. So a state $\Psi\in\mathcal{H}_k$ belongs to $\mathcal{H}_{1,k}$ if and only if it is invariant in the sense that

$$\Psi\big(z_1u^{\mathfrak{p}_1},\dots,z_nu^{\mathfrak{p}_n}\big)=u^k\Psi(z_1,\dots,z_n).$$

Hence there is a holomorphic line orbi-bundle $L_r\to M_r$ such that $\mathcal{H}_{1,k}$ identifies with the space $\mathcal{H}_{r,k}$ of holomorphic sections of L_r^k. L_r has a natural Hermitian structure and connection of curvature $-i\omega_r$, which turns it into a prequantization orbi-bundle (cf. Section 3). We denote by V_k the isomorphism from $\mathcal{H}_{1,k}$ to $\mathcal{H}_{r,k}$.

2.3. Reduction of the operators

On the Bargmann space a usual way to define operators is the Wick or Toeplitz quantization. Denote by Π_k the orthogonal projector of $L^2(\mathbb{C}^n,e^{-k|z|^2}|dz.d\bar{z}|)$ onto $\mathcal{H}_k$. To every function f of $\mathbb{C}^n$ we associate the operator $\mathrm{Op}(f)$ of $\mathcal{H}_k$ defined by

$$\mathrm{Op}(f):\Psi\to\Pi_k(f.\Psi).$$

More generally, we consider multiplicators f which depend on k. Define the class $S(\mathbb{C}^n)$ of symbols $f(\cdot,k)$ which are sequences of $C^\infty(\mathbb{C}^n)$ satisfying:

- there exists $C>0$ and N such that $|f(z,k)|\leqslant C(1+|z|)^N$, $\forall z\in\mathbb{C}^n$, $\forall k$;
- $f(\cdot,k)$ admits an asymptotic expansion of the form

$$\sum_{l=0}^{\infty}k^{-l}f_l+O\big(k^{-\infty}\big)$$

with $f_0,f_1,\ldots\in C^\infty(\mathbb{C}^n)$ for the C^∞ topology on a neighborhood of P.

For such a symbol, we consider $\mathrm{Op}(f(\cdot,k))$ as an unbounded operator of $\mathcal{H}_k$ with domain polynomials on $\mathbb{C}^n$. Its *principal symbol* is the function f_0. If $f(\cdot,k)$ is invariant with respect to the Hamiltonian flow of H, then $\mathrm{Op}(f(\cdot,k))$ sends $\mathcal{H}_{1,k}$ into itself.

Remark 2.1. The class of Toeplitz operators with symbol in $S(\mathbb{C}^n)$ contains the algebra of differential operators generated by $\frac{1}{k}\partial_{z_i}$ and z_i. Indeed, let

$$f(\cdot,k)=P_0+k^{-1}P_1+\cdots+k^{-M}P_M,$$

where $P_0(\bar{z},z),\ldots,P_M(\bar{z},z)$ are polynomials of $\mathbb{C}[\bar{z},z]$. Then $\mathrm{Op}(f(\cdot,k))$ is the operator

$$P_0\left(\frac{1}{k}\partial_z,z\right)+k^{-1}P_1\left(\frac{1}{k}\partial_z,z\right)+\cdots+k^{-M}P_M\left(\frac{1}{k}\partial_z,z\right).$$

Its principal symbol is P_0. If the P_i are linear combinations of the monomials $\bar{z}^\alpha z^\beta$ such that $\langle\mathfrak{p},\alpha-\beta\rangle=0$, then $f(\cdot,k)$ Poisson commutes with H and $\mathrm{Op}(f(\cdot,k))$ preserves the eigenspaces of the quantum harmonic oscillator.

Let U_k be the unitary map $\mathrm{V}_k(\mathrm{V}_k^*\mathrm{V}_k)^{-1/2}:\mathcal{H}_{1,k}\to\mathcal{H}_{r,k}$, that we extend to $\mathcal{H}_k$ in such a way that it vanishes on the orthogonal space to $\mathcal{H}_{1,k}$. The *reduced* operator of $\mathrm{Op}(f(\cdot,k))$ is the operator

$$\mathrm{U}_k\,\mathrm{Op}\big(f(\cdot,k)\big)\mathrm{U}_k^*:\mathcal{H}_{r,k}\to\mathcal{H}_{r,k}.$$

Our main result says it is a Toeplitz operator.

Theorem 2.2. *Let $f(\cdot,k)$ be a symbol of $S(\mathbb{C}^n)$. Then there exists a sequence $g(\cdot,k)$ of $C^\infty(M_r)$, which admits an asymptotic expansion of the form $\sum_{l=0}^\infty k^{-l}g_l+O(k^{-\infty})$ for the C^∞ topology, such that*

$$\mathrm{U}_k\,\mathrm{Op}\big(f(\cdot,k)\big)\mathrm{U}_k^*=\Pi_{r,k}g(\cdot,k)+O\big(k^{-\infty}\big),$$

where $\Pi_{r,k}$ is the orthogonal projector onto $\mathcal{H}_{r,k}$ and the $O(k^{-\infty})$ is for the uniform norm. Furthermore, the principal symbol g_0 of the reduced operator is given by

$$g_0\big(p(x)\big)=\int_{S^1}f_0(l_\theta.x)\frac{|d\theta|}{2\pi},\quad \forall x\in P,$$

where p is the projection $P\to M_r$ and f_0 is the principal symbol of $\mathrm{Op}(f(\cdot,k))$.

2.4. Spectral density

Consider a self-adjoint Toeplitz operator $(\mathrm{T}_k)_k$ of M_r,

$$\mathrm{T}_k=\Pi_{r,k}g(\cdot,k)+O\big(k^{-\infty}\big):\mathcal{H}_{r,k}\to\mathcal{H}_{r,k},$$

where $g(\cdot,k)$ is a sequence of $C^\infty(M_r,\mathbb{R})$ with an asymptotic expansion of the form $\sum_{l=0}^\infty k^{-l}g_l+O(k^{-\infty})$ in the C^∞ topology. Let d_k be the dimension of $\mathcal{H}_{r,k}$ and

$$\lambda_1(k)\leqslant\lambda_2(k)\leqslant\cdots\leqslant\lambda_{d_k}(k)$$

be the eigenvalues of T_k counted with multiplicity.

Let $f\in C^\infty(\mathbb{R})$. The estimate of $\sum_{i=1}^{d_k}f(\lambda_i(k))$ as $k\to\infty$ is a standard semi-classical result when M_r is a manifold. In the orbifold case, this result involves the singular locus of M_r. Denote by G the set of $\zeta=e^{i\theta}$ such that

$$P_\zeta:=\{z\in P;\ l_\theta.z=z\}$$

is not empty. A straightforward computation leads to

$$G=\big\{\zeta\in\mathbb{C}^*;\ \zeta^{\mathfrak{p}_i}=1\text{ for some }i\big\}$$

and

$$P_\zeta=\mathbb{C}_\zeta\cap P\quad\text{with }\mathbb{C}_\zeta=\big\{z\in\mathbb{C}^n;\ z_i=0\text{ if }\zeta^{p_i}\neq1\big\}.$$

The Hamiltonian flow of H preserves P_ζ. Let M_ζ be the quotient of P_ζ by the induced S^1-action. It is a twisted projective space which embeds into M_r as a symplectic suborbifold. Denote by $n(\zeta)$ its complex dimension. Finally, let $m(\zeta)$ be the greatest common divisor of $\{\mathfrak{p}_i;\, \zeta^{\mathfrak{p}_i} = 1\}$.

Theorem 2.3. *For every function $f \in C^\infty(\mathbb{R})$,*

$$\sum_{i=1}^{d_k} f\big(\lambda_i(k)\big) = \sum_{\zeta\in G} \left(\frac{k}{2\pi}\right)^{n(\zeta)} \zeta^{-k} \sum_{l=0}^{\infty} k^{-l} I_l(\zeta) + O\big(k^{-\infty}\big).$$

The leading coefficients are given by

$$I_0(\zeta) = \frac{1}{m(\zeta)} \left(\prod_{i;\zeta^{\mathfrak{p}_i}\neq 1} \big(1-\zeta^{p_i}\big)^{-1} \right) \int_{M_\zeta} f(g_0)\delta_{M_\zeta},$$

where δ_{M_ζ} is the Liouville measure of M_ζ.

Observe that $M_1 = M_r$. The other M_ζ are of positive codimension and are the closures of the singular stratas of M_r. Hence at first order, the formula is the same as in the manifold case

$$\sum_{i=1}^{d_k} f(\lambda_i) = \left(\frac{k}{2\pi}\right)^{n-1} \int_{M_r} f(g_0)\delta_{M_r} + O\big(k^{n-2}\big).$$

Furthermore, applying this result with $f \equiv 1$, we obtain an estimate of the dimension of $\mathcal{H}_k$. When k is sufficiently large, this dimension is also given by the Riemann–Roch–Kawasaki theorem and both results are in agreement (cf. Remark 6.11).

3. The Guillemin–Sternberg isomorphism

Let M be a compact connected Kähler manifold. Denote by $\omega \in \Omega^2(M,\mathbb{R})$ the fundamental two-form. Assume that M is endowed with a prequantization bundle $L \to M$, that is L is a Hermitian line bundle with a connection of curvature $-i\omega$. (M,ω) is a symplectic manifold and represents the classical phase space. For every positive integer k define the quantum space $\mathcal{H}_k$ as the space of holomorphic sections of $L^k \to M$.

Assume that M is endowed with an effective Hamiltonian torus action

$$\mathbb{T}^d \times M \to M, \quad \theta, x \to l_\theta.x \tag{4}$$

which preserves the complex structure. Let $\mathfrak{t}_d$ be the Lie algebra of $\mathbb{T}^d$. If $\xi \in \mathfrak{t}_d$, we denote by $\xi^\#$ the associated vector field of M. Let

$$\mu : M \to \mathfrak{t}_d^*$$

be the moment map, so $\omega(\xi^\#,\cdot) + d\langle\mu,\xi\rangle = 0$.

Following Kostant and Souriau, for every $\xi \in \mathfrak{t}_d$ and every positive integer k we define the operator $\mathrm{M}_{\xi,k}$:

$$\mathrm{M}_{\xi,k} := \langle \mu, \xi \rangle + \frac{1}{ik} \nabla_{\xi^\#} : \mathcal{H}_k \to \mathcal{H}_k.$$

It has to be considered as the quantization of the classical observable $\langle \mu, \xi \rangle \in C^\infty(M)$. Since the Poisson bracket of $\langle \mu, \xi \rangle$ and $\langle \mu, \xi' \rangle$ vanishes, one proves that $\mathrm{M}_{\xi,k}$ commutes with $\mathrm{M}_{\xi',k}$. The *joint spectrum* of the $\mathrm{M}_{\xi,k}$ is the set of covectors $\lambda \in \mathfrak{t}_d^*$ such that

$$\mathcal{H}_{\lambda,k} := \left\{ \Psi \in \mathcal{H}_k;\ \mathrm{M}_{\xi,k} \Psi = \langle \lambda, \xi \rangle \Psi, \forall \xi \in \mathfrak{t}_d \right\}$$

is not reduced to (0).

The joint eigenvalues are related to the values of μ in the following way. First, recall the convexity theorem of Atiyah [1] and Guillemin, Sternberg [12]: the image under μ of the fixed point set of M is a finite set

$$\{\nu_1, \dots, \nu_s\}$$

and $\mu(M)$ is the convex hull of this set.

Theorem 3.1. *Let ν be a value of μ at some fixed point. Let $(\lambda, k) \in \mathfrak{t}_d^* \times \mathbb{N}^*$. Then λ belongs to the joint spectrum of the $\mathrm{M}_{\xi,k}$ only if*

$$\lambda \in \mu(M) \cap \left(\nu + \frac{2\pi}{k} K \right), \tag{5}$$

where K is the integer lattice of $\mathfrak{t}_d^*$.

Condition (5) does not depend on the choice of ν: since (M, ω) is endowed with a prequantization bundle, it is known that for every i, j

$$\nu_i - \nu_j \in 2\pi K. \tag{6}$$

That $\lambda \in \mu(M)$ is necessary has been proved by Guillemin and Sternberg (cf. [13, Theorem 5.3]). The second condition, $\lambda \in \nu + 2\pi k^{-1} K$, is an exact Bohr–Sommerfeld condition, which follows from the theory of Kostant and Souriau.

Example 3.2. Let M be the projective space $\mathbb{CP}^3$ with ω the Fubiny–Study form

$$\omega = -i \partial \bar{\partial} \left(|z_1|^2 + \cdots + |z_4|^2 \right), \quad [z_1, \dots, z_4] \in \mathbb{CP}^3,$$

and L the tautological bundle. Consider the torus action

$$(\theta_1, \theta_2), [z_1, \dots, z_4] \to \left[z_1 e^{-2i\pi(\theta_1+\theta_2)}, z_2 e^{-6i\pi\theta_1}, z_3 e^{-6i\pi\theta_2}, z_4 \right]$$

with momentum map $\mu = \frac{2\pi}{|z|^2}(|z_1|^2 + 3|z_2|^2, |z_1|^2 + 3|z_3|^2)$. The points on Fig. 1 are the λ satisfying condition (5) with $k = 4$ on the left and $k = 2$ on the right. The lines are the critical values of μ.

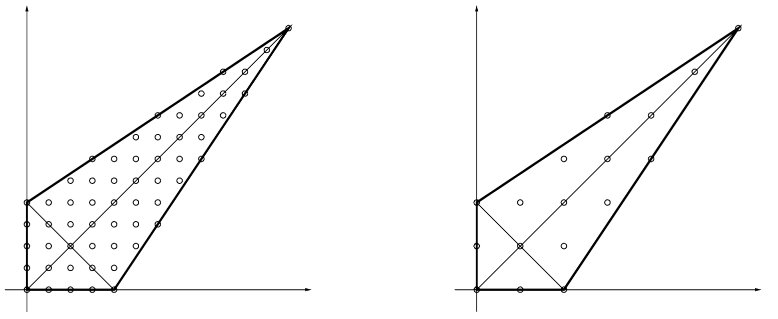

Fig. 1.

Let $(\lambda, k) \in \mathfrak{t}_d^* \times \mathbb{N}^*$. Assume that (λ, k) satisfies condition (5) and that λ is a regular value of μ. Denote by P the level set $\mu^{-1}(\lambda)$. It is known that P is connected [1,12]. The torus action restricts to a locally free action on P. So the quotient M_r of P is a compact connected orbifold. It is naturally endowed with a symplectic form ω_r.

Theorem 3.3. (Guillemin–Sternberg [13]) *M_r inherits by reduction a Kähler structure with fundamental 2-form ω_r and a prequantization orbi-bundle $L_r^k \to M_r$ with curvature $-ik\omega_r$. Furthermore, there exists a natural isomorphism of vector space*

$$\mathrm{V}_k : \mathcal{H}_{\lambda,k} \to \mathcal{H}_{r,k},$$

where $\mathcal{H}_{r,k}$ is the space of holomorphic sections of L_r^k.

Now consider a fixed regular value λ of μ such that the set of integers k satisfying condition (5) is not empty. Then there exists a positive integer κ such that (λ, k) satisfies (5) if and only if k is a positive multiple of κ. Furthermore, the Kähler structure of M_r does not depend on k and for every such k,

$$L_r^k = \left(L_r^\kappa\right)^{\otimes k/\kappa}.$$

If M_r is a manifold, it follows from Kodaira vanishing theorem and Riemann–Roch theorem that

$$\dim \mathcal{H}_{r,k} = \left(\frac{k}{2\pi}\right)^{n_r} \mathrm{Vol}(M_r) + O\left(k^{n_r - 1}\right) \tag{7}$$

as k goes to infinity, where $\mathrm{Vol}(M_r)$ is the symplectic volume of M_r and n_r its dimension. This gives a partial converse to Theorem 3.1: if k is a sufficiently large positive multiple of κ, the eigenspace $\mathcal{H}_{\lambda,k}$ is not reduced to (0), so λ belongs to the joint spectrum of the $M_{\xi,k}$.

If M_r is an orbifold, the same result holds and follows from Riemann–Roch–Kawasaki theorem [17]. Indeed, we assumed that the torus action is effective. This implies that its restriction to P is also effective. So M_r is a reduced orbifold or equivalently its principal stratum has multiplicity one. This explains why formula (7) remains unchanged, without a sum of oscillatory terms.

All the semi-classical results we prove in this paper are in this regime,

$$\lambda \textit{ is fixed and } k \to \infty \textit{ running through the set of positive multiples of } \kappa.$$

In the remainder of this section, we recall the main steps of the proof of the Guillemin–Sternberg theorem. We follow the presentation given by Duistermaat in [9], that is we consider separately the reduction of the symplectic and prequantum data and the reduction of the complex structure. We explain in remarks how the same constructions apply to the harmonic oscillator.

Remark 3.4. (*Harmonic oscillator*) The Bargmann space can be viewed as a space of holomorphic sections of a prequantization bundle over $\mathbb{C}^n$. Let $L := \mathbb{C}^n \times \mathbb{C}$ be the trivial bundle over $\mathbb{C}^n$. We identify the sections of L^k with the functions on $\mathbb{C}^n$. Introduce a connection and a Hermitian structure on L^k by setting

$$\nabla\Psi = d\Psi - k\Psi(\bar{z}_1 dz_1 + \cdots + \bar{z}_n dz_n), \qquad (\Psi, \Psi)(z) = e^{-k|z|^2}\big|\Psi(z)\big|^2.$$

In this way L^k becomes a prequantization bundle with curvature $-ik\omega$. The scalar product defined in (1) is

$$(\Psi, \Psi)_{\mathbb{C}^n} = \int_{\mathbb{C}^n} (\Psi, \Psi)(z) \frac{|dz.d\bar{z}|}{n!}.$$

So the Bargmann space $\mathcal{H}_k$ is the Hilbert space of holomorphic sections Ψ of L^k such that $(\Psi, \Psi)_{\mathbb{C}^n}$ is finite. Furthermore, it is easily checked that the quantum harmonic oscillator H_k defined in (2) is given by

$$\mathrm{H}_k\Psi = \left(H + \frac{1}{ik}\nabla_{X_H}\right)\Psi,$$

where X_H is the Hamiltonian vector field of H.

3.1. Reduction of the symplectic and prequantum data

We first lift the torus action (4). If $z \in L_x^k$ and $\xi \in \mathfrak{t}_d$, we denote by $\mathcal{T}_\xi.z$ the parallel transport of z along the path

$$[0, 1] \to M, \quad s \to l_{\exp(s\xi)}.x.$$

If ν is the value of μ at some fixed point and ξ belong to the integer lattice of $\mathfrak{t}_d$,

$$e^{ik\langle\nu-\mu,\xi\rangle}\mathcal{T}_\xi.z = z \quad \text{for every } z \in L^k.$$

Indeed, this is obviously true if $z \in L_x^k$ where x is a fixed point and $\mu(x) = \nu$. By [10, Proposition 15.3], the result follows for every z.

Consider now $(\lambda, k) \in \mathfrak{t}_d^* \times \mathbb{N}^*$ which satisfies condition (5). Then the action of $\mathbb{T}^d$ on M lifts to L^k

$$\mathbb{T}^d \times L^k \to L^k, \quad (\theta, z) \to \mathcal{L}_\theta.z := e^{ik\langle\lambda-\mu,\xi\rangle}\mathcal{T}_\xi.z$$

with $\xi \in \mathfrak{t}_d$ such that $\exp \xi = \theta$. One can check the following facts: for every θ, $\mathcal{L}_\theta$ is an automorphism of the prequantization bundle L^k, it preserves the complex structure. Furthermore, the obtained representation of $\mathbb{T}^d$ on $\mathcal{H}_k$ induces the representation of the Lie algebra $\mathfrak{t}_d$ given by the operators

$$\nabla_{\xi^\#} + ik\langle \mu - \lambda, \xi \rangle = ik\bigl(\mathrm{M}_{\xi,k} - \langle \lambda, \xi \rangle\bigr), \quad \xi \in \mathfrak{t}_d.$$

Hence the joint eigenspace $\mathcal{H}_{\lambda,k}$ is the space of invariant holomorphic sections

$$\mathcal{H}_{\lambda,k} = \bigl\{\Psi \in \mathcal{H}_k;\ \mathcal{L}_\theta^* \Psi = \Psi,\ \forall \theta \in \mathbb{T}^d\bigr\}.$$

Denote by $j : P \to M$ and $p : P \to M_r$ the natural embedding and projection. Recall that the reduced symplectic 2-form ω_r is defined by $p^*\omega_r = j^*\omega$. Let L_r^k be the quotient of j^*L^k by the torus action. This is a Hermitian orbi-bundle over M_r and $p^*L_r^k$ is naturally isomorphic with j^*L^k. Furthermore, L_r^k admits a connection ∇ such that

$$p^*\nabla = j^*\nabla.$$

Its curvature is $-ik\omega_r$. So L_r^k is a prequantization orbi-bundle. Since the sections of $\mathcal{H}_{\lambda,k}$ are invariant, their restrictions to P descend to M_r,

$$\mathcal{H}_{\lambda,k} \to C^\infty\bigl(M_r, L_r^k\bigr), \quad \Psi \to \Psi_r \quad \text{such that } p^*\Psi_r = j^*\Psi. \tag{8}$$

This is the first definition of the Guillemin–Sternberg isomorphism. In the case the action is not free and M_r is an orbifold, more details will be given in Remark 3.8.

Remark 3.5. (*Harmonic oscillator*) As in Section 2, we only consider the eigenvalue $\lambda = 1$. The lift of the S^1-action is explicitly given by

$$S^1 \times \bigl(\mathbb{C}^n \times \mathbb{C}\bigr) \to \mathbb{C}^n \times \mathbb{C}, \quad \theta,\ (z_1, \dots, z_n, v) \to \bigl(e^{i\mathfrak{p}_1\theta} z_1, \dots, e^{i\mathfrak{p}_n\theta} z_n, e^{ik\theta} v\bigr).$$

$\mathcal{H}_{1,k}$ consists of the invariant holomorphic sections of L^k. If Ψ is such a section, one can check by direct computations that (Ψ, Ψ) and $\nabla\Psi$ are invariant and that $\nabla_{X_H}\Psi$ vanishes over the level set $P := \{H = 1\}$. So the quotient L_r^k of $L^k|_P$ by S^1 inherits a structure of prequantization bundle with curvature $-ik\omega_r$.

3.2. Complex reduction

Let $\mathbb{T}^d_{\mathbb{C}}$ be the complex Lie group $\mathbb{T}^d \oplus i\mathfrak{t}_d$ with Lie algebra $\mathfrak{t}_d \oplus i\mathfrak{t}_d$. We consider $\mathbb{T}^d$ as a subgroup of $\mathbb{T}^d_{\mathbb{C}}$. Since the torus action preserves the complex structure of M, it can be extended in a unique way to a holomorphic action of $\mathbb{T}^d_{\mathbb{C}}$:

$$\bigl(\mathbb{T}^d \oplus i\mathfrak{t}_d\bigr) \times M \to M, \quad (\theta + it),\ x \to l_{\theta+it}.x.$$

To do this, for every $\xi \in \mathfrak{t}_d$, we define the infinitesimal generator of $i\xi$ as $J\xi^\#$, where J is the complex structure of M. Since M is compact, we can integrate $J\xi^\#$. Then one can check that this defines a holomorphic action.

In a similar way, the action on L^k extends to a holomorphic action

$$\left(\mathbb{T}^d \oplus i\mathfrak{t}_d\right) \times L^k \to L^k, \quad (\theta + it), z \to \mathcal{L}_{\theta+it}.z,$$

where $\mathcal{L}_{\theta+it}$ is an automorphism of complex bundle which lifts $l_{\theta+it}$. This gives a representation of $\mathbb{T}^d_{\mathbb{C}}$ on $\mathcal{H}_k$. The induced representation of the Lie algebra $\mathfrak{t}_d \oplus i\mathfrak{t}_d$ is given by the operators

$$\nabla_{\xi^\# + J\eta^\#} + ik\langle \mu - \lambda, \xi + i\eta\rangle, \quad \xi + i\eta \in \mathfrak{t}_d \oplus i\mathfrak{t}_d. \tag{9}$$

Furthermore $\mathcal{H}_{\lambda,k}$ is the space of $\mathbb{T}^d_{\mathbb{C}}$-invariant holomorphic sections.

Remark 3.6. (*Harmonic oscillator*) The holomorphic action of $\mathbb{C}^*$ was given in (3). It lifts to

$$\mathbb{C}^* \times \left(\mathbb{C}^n \times \mathbb{C}\right) \to \mathbb{C}^n \times \mathbb{C}, \quad u, (z_1, \dots, z_n, v) \to \left(u^{\mathfrak{p}_1} z_1, \dots, u^{\mathfrak{p}_n} z_n, u^k v\right).$$

The invariant sections of $\mathcal{H}_k$ are obviously the sections of $\mathcal{H}_{1,k}$.

Let $P_{\mathbb{C}}$ be the saturated set $\mathbb{T}^d_{\mathbb{C}}.P$ of P. It is an open set of M. The next step is to consider the quotient of $P_{\mathbb{C}}$ by $\mathbb{T}^d_{\mathbb{C}}$. This have to be done carefully because the $\mathbb{T}^d_{\mathbb{C}}$-action on $P_{\mathbb{C}}$ is not proper. Actually, the map

$$\mathfrak{t}_d \times P \to P_{\mathbb{C}}, \quad t, y \to l_{it}.y, \tag{10}$$

is a diffeomorphism. So every $\mathbb{T}^d_{\mathbb{C}}$-orbit of $P_{\mathbb{C}}$ intersects P in a $\mathbb{T}^d$-orbit and the injection $P \to P_{\mathbb{C}}$ induces a bijection from M_r onto $P_{\mathbb{C}}/\mathbb{T}^d_{\mathbb{C}}$. Furthermore, every slice $U \subset P$ for the $\mathbb{T}^d$-action on P is a slice for the $\mathbb{T}^d_{\mathbb{C}}$-action. Viewed as a quotient by a holomorphic action, the orbifold M_r inherits a complex structure. This complex structure is compatible with ω_r.

Similarly, the bundle L^k_r may be considered as the quotient of $L^k|_{P_{\mathbb{C}}}$ by the complex action and inherits a holomorphic structure. This is the unique holomorphic structure compatible with the connection and the Hermitian product. Denote by $p_{\mathbb{C}}$ the projection $P_{\mathbb{C}} \to M_r$ and observe that $p^*_{\mathbb{C}} L^k_r$ is naturally isomorphic with $L^k|_{P_{\mathbb{C}}}$.

The interest of viewing L^k_r and M_r as complex quotients is that there is a natural identification of the $\mathbb{T}^d_{\mathbb{C}}$-invariant holomorphic sections Ψ of $L^k \to P_{\mathbb{C}}$ with the holomorphic sections Ψ_r of $L^k_r \to M_r$, given by $p^*_{\mathbb{C}}\Psi_r = \Psi$. So the map (8) takes its values in the space $\mathcal{H}_{r,k}$ of holomorphic sections of L^k_r.

Definition 3.7. $\mathrm{V}_k : \mathcal{H}_{\lambda,k} \to \mathcal{H}_{r,k}$ is the map which sends Ψ into the section Ψ_r such that

$$p^*_{\mathbb{C}}\Psi_r = \left(\frac{2\pi}{k}\right)^{d/4} \Psi|_{P_{\mathbb{C}}} \quad \text{or equivalently} \quad p^*\Psi_r = \left(\frac{2\pi}{k}\right)^{d/4} j^*\Psi.$$

The rescaling by $(2\pi/k)^{d/4}$ is such that V_k and its inverse are bounded independently of k (cf. Proposition 4.22).

Remark 3.8. (*Orbifold*) Let us detail the previous constructions when the $\mathbb{T}^d$-action is not free. For the basic definitions of the theory of orbifolds, our references are [10, Section 14.1] and [8, the appendix].

As topological spaces, M_r and L_r^k are the quotients $P/\mathbb{T}^d$ and $j^*L^k/\mathbb{T}^d$. M_r is naturally endowed with a collection of orbifold charts in the following way. Let $x \in P$, $G \subset \mathbb{T}^d$ be its isotropy subgroup and $U \subset P$ be a slice at x for the $\mathbb{T}^d$-action. Denote by π_U the projection $U \to M_r$ and by $|U| \subset M_r$ its image. Then $(|U|, U, G, \pi_U)$ is an orbifold chart of M_r, i.e. $|U|$ is an open set of M_r, U a manifold, G a finite group which acts on U by diffeomorphisms and π_U factors through a homeomorphism $U/G \to |U|$. These charts cover M_r and satisfy some compatibility conditions, which defines the orbifold structure of M_r.

For every such chart, the bundle L^k restricts to a G-bundle

$$L^k_{r,U} \to U.$$

These bundles are orbifold charts of the orbi-bundle $L_r^k \to M_r$. A section of L_r^k is a continuous section of $L_r^k \to M_r$ which lifts to a G-invariant C^∞ section of $L^k_{r,U}$ for every U. Since every $\mathbb{T}^d$-invariant section of L^k restricts to a G-invariant section of $L^k_{r,U}$, the map (8) is well defined. Continuing in this way, we can introduce the Kähler structure of M_r, the Hermitian and holomorphic structures of L_r^k, its connection and verify that we obtain a well defined map V_k as in Definition 3.7.

It is also useful to consider P and $P_{\mathbb{C}}$ as orbifolds and the projections $p: P \to M_r$ and $p_{\mathbb{C}}: P_{\mathbb{C}} \to M_r$ as orbifold maps. For instance, let $(|U|, U, G, \pi_U)$ be a chart defined as above. Let

$$V := \mathbb{T}^d \times \mathfrak{t}_d \times U, \qquad |V| := \mathbb{T}^d_{\mathbb{C}}.U,$$

and π_V be the map $V \to |V|$ which sends (θ, t, u) into $l_{\theta + it}.u$. Let G acts on V by

$$G \times V \to V, \quad g,\ (\theta, t, u) \to (\theta - g, t, l_g.u).$$

Then $(|V|, V, G, \pi_V)$ is an orbifold chart of $P_{\mathbb{C}}$. Furthermore, $\mathbb{T}^d_{\mathbb{C}}$ acts on $V = \mathbb{T}^d_{\mathbb{C}} \times U$ by left multiplication, this action lifts the $\mathbb{T}^d_{\mathbb{C}}$-action on $|V|$, and the projection $V \to U$ locally lifts $p_{\mathbb{C}}$:

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\pi_V"] & U \arrow[d, "\pi_U"] \\ {|V|} \arrow[r] & {|U|.} \end{tikzcd} \tag{11}$$

Now, instead of viewing U as a submanifold of P, we consider it as the quotient of V by $\mathbb{T}^d_{\mathbb{C}}$. To define the various structure on U, we can lift everything from $|V|$ to V and we perform the reduction from V to U. Since the $\mathbb{T}^d_{\mathbb{C}}$-action on V is free, we are reduced to the manifold case. Furthermore, since $V \to |V|$ is a G-principal bundle and $V \to U$ is G-equivariant, we obtain G-invariant structures. We can apply the same method with the map $p: P \to M_r$.

Remark 3.9. (*Proof of Theorem 3.3*) Since $P_{\mathbb{C}}$ is open, V_k is injective. That V_k is surjective is more difficult to prove. It consists to show that every invariant holomorphic section of $L^k \to P_{\mathbb{C}}$ extends to an invariant holomorphic section over M. Let us precise that the proof of Guillemin and Sternberg extends to the orbifold case without modification. The only technical point is to show that there exists a non-vanishing section in $\mathcal{H}_{\lambda,k}$, when k is sufficiently large [13, Theorem 5.6]. This will be proved in Section 7.2, cf. Remark 7.5.

Remark 3.10. (*Harmonic oscillator*) The definition of $\mathrm{V}_k : \mathcal{H}_{1,k} \to \mathcal{H}_{r,k}$ is the same. It is easily checked that this map is onto: every holomorphic section of L_r^k lifts to a holomorphic section Ψ of L^k over $\mathbb{C}^n - \{0\}$ satisfying

$$\Psi\left(u^{\mathfrak{p}_1} z_1, \dots, u^{\mathfrak{p}_n} z_n\right) = u^k \Psi(z_1, \dots, z_n). \tag{12}$$

Since it is bounded on a neighborhood of the origin, it extends on $\mathbb{C}^n$. Writing its Taylor expansion at the origin, we deduce from (12) that Ψ is polynomial and belongs to $\mathcal{H}_{1,k}$.

4. Reduction of Toeplitz operators

4.1. Toeplitz operators

Let us denote by $L^2(M, L^k)$ the space of L^2 sections of L^k. We define the scalar product of sections of L^k as

$$(\Psi, \Psi')_M = \int_M (\Psi, \Psi')(x) \delta_M(x),$$

where (Ψ, Ψ') is the punctual scalar product and δ_M is the Liouville measure $\frac{1}{n!}|\omega^{\wedge n}|$. Let Π_k be the orthogonal projector of $L^2(M, L^k)$ onto $\mathcal{H}_k$.

Given $f \in C^\infty(M)$, we denote by M_f the operator of $L^2(M, L^k)$ sending Ψ into $f\Psi$. The set of symbols $S(M)$ consists of the sequences $(f(\cdot, k))_k$ of $C^\infty(M)$ which admit an asymptotic expansion of the form

$$f(\cdot, k) = \sum_{l=0}^{\infty} k^{-l} f_l + O\left(k^{-\infty}\right), \quad \text{with } f_0, f_1, \dots \in C^\infty(M), \tag{13}$$

for the C^∞ topology.

A Toeplitz operator is a family $(\mathrm{T}_k)_k$ of the form

$$\mathrm{T}_k = \Pi_k M_{f(\cdot,k)} \Pi_k + \mathrm{R}_k, \tag{14}$$

where $(f(\cdot, k)) \in S(M)$ and R_k is an operator of $L^2(M, L^k)$ satisfying $\Pi_k \mathrm{R}_k \Pi_k = \mathrm{R}_k$ and whose uniform norm is $O(k^{-\infty})$. The following result is a consequence of the works of Boutet de Monvel and Guillemin [4] (cf. [6]).

Theorem 4.1. *The set $\mathcal{T}$ of Toeplitz operators is a $*$-algebra. The contravariant symbol map*

$$\sigma_{\mathrm{cont}} : \mathcal{T} \to C^\infty(M)[\![\hbar]\!], \quad \Pi_k M_{f(\cdot,k)} \Pi_k + \mathrm{R}_k \to \sum \hbar^l f_l,$$

is well defined, onto and its kernel consists of the Toeplitz operators whose uniform norm is $O(k^{-\infty})$. Furthermore, the product $*_c$ induced on $C^\infty(M)[\![\hbar]\!]$ is a star-product.

The *principal* symbol of a Toeplitz operator is the first coefficient f_0 of its contravariant symbol. The operators $\mathrm{M}_{\xi,k}$ are Toeplitz operators with principal symbol $\langle \mu, \xi \rangle$.

We use the same definitions and notations over M_r. Recall that λ is fixed and k runs over the positive multiples of κ. So a Toeplitz operator of $\mathcal{T}_r$ is a family

$$(\mathrm{T}_k)_{k=\kappa,2\kappa,\dots}.$$

We denote by $\Pi_{r,k}$ the orthogonal projector onto $\mathcal{H}_{r,k}$ and by $*_{cr}$ the product of the contravariant symbols of $C^\infty(M_r)[\![\hbar]\!]$.

Remark 4.2. (*Orbifold*) In the case M_r is an orbifold, the definition of the Toeplitz operators makes sense. We will prove Theorem 4.1 for the Toeplitz operators of M_r in Section 6.

Remark 4.3. (*Harmonic oscillator*) To avoid a discussion about the infinity of $\mathbb{C}^n$, we do not define the full algebra of Toeplitz operators on $\mathbb{C}^n$ and do not state any result similar to Theorem 4.1. We only consider the Toeplitz operators of the form

$$\Pi_k M_{f(\cdot,k)} \Pi_k,$$

where $f(\cdot,k)$ is a symbol of $S(\mathbb{C}^n)$ (cf. definition in Section 2.3).

4.2. Statement of the main result

Recall that V_k is the isomorphism from $\mathcal{H}_{\lambda,k}$ to $\mathcal{H}_{r,k}$ (cf. Definition 3.7). Let U_k be the operator

$$L^2(M, L^k) \to L^2(M_r, L_r^k), \quad \Psi \to \begin{cases} \mathrm{V}_k(\mathrm{V}_k^*\mathrm{V}_k)^{-1/2}\Psi, & \text{if } \Psi \in \mathcal{H}_{\lambda,k}, \\ 0, & \text{if } \Psi \text{ is orthogonal to } \mathcal{H}_{\lambda,k}. \end{cases}$$

Hence

$$\mathrm{U}_k^*\mathrm{U}_k = \Pi_{\lambda,k}, \qquad \mathrm{U}_k\mathrm{U}_k^* = \Pi_{r,k}, \qquad \Pi_{r,k}\mathrm{U}_k\Pi_{\lambda,k} = \mathrm{U}_k,$$

where $\Pi_{\lambda,k}$ is the orthogonal projector onto $\mathcal{H}_{\lambda,k}$. The main result of the section is the following theorem and the corresponding Theorem 2.2 for the harmonic oscillator.

Theorem 4.4. *Let* T_k *be a Toeplitz operator of* M *with principal symbol* f. *Then*

$$\mathrm{U}_k\mathrm{T}_k\mathrm{U}_k^* : L^2(M_r, L_r^k) \to L^2(M_r, L_r^k)$$

is a Toeplitz operator of M_r. *Its principal symbol is the function* $g \in C^\infty(M_r)$ *such that*

$$g(p(x)) = \int_{\mathbb{T}^d} f(l_\theta.x)\delta_{\mathbb{T}^d}(\theta), \quad x \in P,$$

with $\delta_{\mathbb{T}^d}$ *the Haar measure of* $\mathbb{T}^d$.

In the following subsection we introduce the λ-Toeplitz operators. These are the operators of the form

$$\Pi_{\lambda,k} \mathrm{T}_k \Pi_{\lambda,k}, \quad \text{where } \mathrm{T}_k \in \mathcal{T}.$$

In the next subsections, we prove some estimates for the sections of $\mathcal{H}_{\lambda,k}$ and introduce an integration map. Then we prove that the space of λ-Toeplitz operators is isomorphic to the space of Toeplitz operators of M_r, a stronger result than Theorem 4.4.

Our proof uses the properties of the Toeplitz operators of M_r stated in Theorem 4.1. So, in the case the $\mathbb{T}^d$-action is not free, the proof will be complete only in Section 6 with the proof of Theorem 4.1 for orbifolds.

For the following, we introduce the inverse $\mathrm{W}_k : \mathcal{H}_{r,k} \to \mathcal{H}_{\lambda,k}$ of V_k. We consider that V_k and W_k act not only on $\mathcal{H}_{\lambda,k}$ and $\mathcal{H}_{r,k}$, respectively, but on the space of L^2 sections in such a way that they vanish on the orthogonal of $\mathcal{H}_{\lambda,k}$ and $\mathcal{H}_{r,k}$, respectively. So

$$\Pi_{r,k} \mathrm{V}_k \Pi_{\lambda,k} = \mathrm{V}_k, \qquad \Pi_{\lambda,k} \mathrm{W}_k \Pi_{r,k} = \mathrm{W}_k, \qquad \mathrm{V}_k \mathrm{W}_k = \Pi_{r,k}, \qquad \mathrm{W}_k \mathrm{V}_k = \Pi_{\lambda,k},$$

and with the convention $0^{-1/2} = 0$, the equality $\mathrm{U}_k = \mathrm{V}_k (\mathrm{V}_k^* \mathrm{V}_k)^{-1/2}$ is valid on $L^2(M, L^k)$. Furthermore, we say that a function or a section is invariant if it is invariant with respect to the action of $\mathbb{T}^d$.

4.3. The λ-Toeplitz operators

We begin with a useful formula for the orthogonal projector $\Pi_{\lambda,k}$ onto $\mathcal{H}_{\lambda,k}$. Denote by $\mathrm{P}_{\lambda,k}$ the orthogonal projector of $L^2(M, L^k)$ onto the space of invariant sections of L^k (not necessarily holomorphic). If Ψ is a section of L^k, $\mathrm{P}_{\lambda,k}\Psi$ is given by the well-known formula

$$\mathrm{P}_{\lambda,k}\Psi = \int_{\mathbb{T}^d} \mathcal{L}_\theta^* \Psi \, \delta_{\mathbb{T}^d}(\theta), \tag{15}$$

where $\delta_{\mathbb{T}^d}$ is the Haar measure. Since $\mathrm{P}_{\lambda,k}$ sends $\mathcal{H}_k$ in $\mathcal{H}_k$, we have

$$\Pi_{\lambda,k} = \mathrm{P}_{\lambda,k} \Pi_k = \Pi_k \mathrm{P}_{\lambda,k}. \tag{16}$$

Let $\mathcal{T}_\lambda$ be the set of λ-Toeplitz operators

$$\mathcal{T}_\lambda := \{ \Pi_{\lambda,k} \mathrm{T}_k \Pi_{\lambda,k};\ \mathrm{T}_k \in \mathcal{T} \}.$$

This first result follows from Theorem 4.1 about Toeplitz operators.

Theorem 4.5. *$\mathcal{T}_\lambda$ is a $*$-algebra. Furthermore, if T_k is a Toeplitz operator with contravariant symbol $\sum \hbar^l f_l$, then*

$$\Pi_{\lambda,k} \mathrm{T}_k \Pi_{\lambda,k} = \Pi_{\lambda,k} M_{f_\lambda(\cdot,k)} \Pi_{\lambda,k} + \mathrm{R}_k, \tag{17}$$

where R_k *is* $O(k^{-\infty})$, $\Pi_{\lambda,k}\mathrm{R}_k\Pi_{\lambda,k} = \mathrm{R}_k$ *and* $f_\lambda(\cdot, k)$ *is an invariant symbol of* M *with an asymptotic expansion* $\sum k^{-l} f_{\lambda,l}$ *such that*

$$f_{\lambda,l}(x) = \int_{\mathbb{T}^d} f_l(l_\theta .x)\delta_{\mathbb{T}^d}(\theta), \quad l = 0, 1, \ldots .$$

So the λ-Toeplitz operators can also be defined as the operators of the form

$$\Pi_{\lambda,k} M_{f(\cdot,k)} \Pi_{\lambda,k} + \mathrm{R}_k, \tag{18}$$

where the multiplicator $f(\cdot, k) \in S(M)$ is invariant, $\Pi_{\lambda,k}\mathrm{R}_k\Pi_{\lambda,k} = \mathrm{R}_k$ and R_k is $O(k^{-\infty})$.

A similar algebra was introduced by Guillemin and Sternberg [14] in the context of pseudodifferential operators. Their main theorem was that there exists an associated symbolic calculus, where the symbols are defined on the reduced space M_r. Let us state the corresponding result in our context.

Theorem 4.6. *The map* $\sigma_{\mathrm{princ}} : \mathcal{T}_\lambda \to C^\infty(M_r)$, *which associates to a* λ*-Toeplitz operator of the form* (18) *with an invariant multiplicator* $f(\cdot, k)$, *the function* $g_0 \in C^\infty(M_r)$ *such that*

$$f(\cdot, k) = p^* g_0 + O\big(k^{-1}\big) \quad \textit{over } P$$

is well defined. Furthermore, the following sequence is exact

$$0 \to \mathcal{T}_\lambda \cap O\big(k^{-1}\big) \to \mathcal{T}_\lambda \xrightarrow{\sigma_{\mathrm{princ}}} C^\infty(M_r) \to 0.$$

Finally if T_k^1 *and* T_k^2 *are* λ*-Toeplitz operators, then*

$$\sigma_{\mathrm{princ}}\big(\mathrm{T}_k^1.\mathrm{T}_k^2\big) = \sigma_{\mathrm{princ}}\big(\mathrm{T}_k^1\big).\sigma_{\mathrm{princ}}\big(\mathrm{T}_k^2\big).$$

So $[\mathrm{T}_k^1, \mathrm{T}_k^2]$ *is* $O(k^{-1})$ *and* $k[\mathrm{T}_k^1, \mathrm{T}_k^2]$ *belongs to* $\mathcal{T}_\lambda$. *Its principal symbol is*

$$\sigma_{\mathrm{princ}}\big(k\big[\mathrm{T}_k^1, \mathrm{T}_k^2\big]\big) = i\big\{\sigma_{\mathrm{princ}}\big(\mathrm{T}_k^1\big), \sigma_{\mathrm{princ}}\big(\mathrm{T}_k^2\big)\big\},$$

where $\{\cdot,\cdot\}$ *is the Poisson bracket of* $C^\infty(M_r)$.

This theorem does not follow from Theorem 4.1. Actually it is a corollary of Theorem 4.25, which says that the algebra of λ-Toeplitz operators and the algebra of Toeplitz operators of M_r are isomorphic (cf. Remark 4.27).

Proof of Theorem 4.5. First let us prove the second point. Let

$$\mathrm{T}_k = \Pi_k M_{f(\cdot,k)} \Pi_k + \mathrm{R}_k$$

be a Toeplitz operator. Since $\Pi_k \Pi_{\lambda,k} = \Pi_{\lambda,k}\Pi_k = \Pi_{\lambda,k}$, we have

$$\Pi_{\lambda,k}\mathrm{T}_k\Pi_{\lambda,k} = \Pi_{\lambda,k} M_{f(\cdot,k)} \Pi_{\lambda,k} + \Pi_{\lambda,k}\mathrm{R}_k\Pi_{\lambda,k}.$$

Clearly R_k is $O(k^{-\infty})$ implies that $\Pi_{\lambda,k}\mathrm{R}_k\Pi_{\lambda,k}$ is $O(k^{-\infty})$. Since $\Pi_{\lambda,k}\mathrm{P}_{\lambda,k} = \mathrm{P}_{\lambda,k}\Pi_{\lambda,k} = \Pi_{\lambda,k}$, we have

$$\Pi_{\lambda,k} M_{f(\cdot,k)} \Pi_{\lambda,k} = \Pi_{\lambda,k}\mathrm{P}_{\lambda,k} M_{f(\cdot,k)} \mathrm{P}_{\lambda,k}\Pi_{\lambda,k}.$$

Then using (15), we obtain

$$\mathrm{P}_{\lambda,k} M_{f(\cdot,k)} \mathrm{P}_{\lambda,k} = M_{f_\lambda(\cdot,k)} \mathrm{P}_{\lambda,k},$$

where $f_\lambda(\cdot, k)$ is the invariant symbol

$$f_\lambda(x,k) = \int_{\mathbb{T}^d} f(l_\theta.x, k)\delta_{\mathbb{T}^d}(\theta).$$

Consequently,

$$\Pi_{\lambda,k}\mathrm{T}_k\Pi_{\lambda,k} = \Pi_{\lambda,k} M_{f_\lambda(\cdot,k)} \Pi_{\lambda,k} + \Pi_{\lambda,k}\mathrm{R}_k\Pi_{\lambda,k},$$

which gives the result. To prove that $\mathcal{T}_\lambda$ is a $*$-algebra, the only difficulty is to check that the product of two λ-Toeplitz operators is a λ-Toeplitz operator. Let $f^1(\cdot, k)$ and $f^2(\cdot, k)$ be invariant symbols of $S(M)$. We have to show that

$$\Pi_{\lambda,k} M_{f^1(\cdot,k)} \Pi_{\lambda,k} M_{f^2(\cdot,k)} \Pi_{\lambda,k}$$

is a λ-Toeplitz operator. By (16),

$$\Pi_{\lambda,k} M_{f^1(\cdot,k)} \Pi_{\lambda,k} M_{f^2(\cdot,k)} \Pi_{\lambda,k} = \Pi_{\lambda,k} M_{f^1(\cdot,k)} \Pi_k \mathrm{P}_{\lambda,k} M_{f^2(\cdot,k)} \Pi_{\lambda,k}.$$

Since $f^2(\cdot, k)$ is invariant, $\mathrm{P}_{\lambda,k}$ and $M_{f^2(\cdot,k)}$ commute, so

$$= \Pi_{\lambda,k} M_{f^1(\cdot,k)} \Pi_k M_{f^2(\cdot,k)} \mathrm{P}_{\lambda,k}\Pi_{\lambda,k} = \Pi_{\lambda,k}\Pi_k M_{f^1(\cdot,k)} \Pi_k M_{f^2(\cdot,k)} \Pi_k\Pi_{\lambda,k}$$

by (16). Finally $\Pi_k M_{f^1(\cdot,k)} \Pi_k M_{f^2(\cdot,k)} \Pi_k$ is a Toeplitz operator since it is the product of two Toeplitz operators. □

Remark 4.7. (*Harmonic oscillator*) We define the λ-Toeplitz operators as the operators of the form

$$\Pi_{1,k} M_{f(\cdot,k)} \Pi_{1,k} + \mathrm{R}_k,$$

where $f(\cdot, k)$ belongs to $S(\mathbb{C}^n)$, $\Pi_{1,k}$ is the orthogonal projector onto $\mathcal{H}_{1,k}$, R_k satisfies $\Pi_{1,k}\mathrm{R}_k\Pi_{1,k} = \mathrm{R}_k$ and its uniform norm is $O(k^{-\infty})$. Then Theorems 4.5 and 4.6 remain true. The proof that the multiplicator $f(\cdot, k)$ can be chosen invariant is the same. The fact that these operators form an algebra and the definition and properties of the principal symbol are consequences of Theorem 4.25.

4.4. Norm of the invariant states

In this section, we estimate the norm of the eigensections of $\mathcal{H}_{\lambda,k}$. We begin with the estimation over $P_{\mathbb{C}}$. Using the diffeomorphism (10), we identify $P_{\mathbb{C}}$ with $\mathfrak{t}_d \times P$. Let ξ_i be a basis of $\mathfrak{t}_d$ and denote by t_i the associated linear coordinates.

Proposition 4.8. *For every $\mathbb{T}^d_{\mathbb{C}}$-invariant section Ψ of $L^k \to P_{\mathbb{C}}$, we have*

$$(\Psi, \Psi)(t, y) = e^{-k\varphi(t,y)}(\Psi, \Psi)(0, y), \quad \forall (t, y) \in \mathfrak{t}_d \times P, \tag{19}$$

where φ is the C^∞ function on $\mathfrak{t}_d \times P$ solution of the equations

$$\varphi(0, y) = 0, \qquad \partial_{t_i}\varphi(t, y) = 2\big(\lambda_i - \mu_i(t, y)\big), \quad \text{with } i = 1, \ldots, d,$$

and $\mu_i := \langle \mu, \xi_i \rangle$, $\lambda_i := \langle \lambda, \xi_i \rangle$ are the components of μ and λ.

Proof. By Eq. (9), $i\xi \in i\mathfrak{t}_d$ acts on the sections of L^k by

$$\nabla_{J\xi^\#} - k\langle \mu - \lambda, \xi \rangle.$$

So if Ψ is a $\mathbb{T}^d_{\mathbb{C}}$-invariant section, then

$$\nabla_{J\xi^\#}\Psi = k\langle \mu - \lambda, \xi \rangle \Psi,$$

which leads to

$$\big(J\xi^\#\big).(\Psi, \Psi) = 2k\langle \mu - \lambda, \xi \rangle (\Psi, \Psi) \tag{20}$$

and shows the proposition. □

On the complementary set $P^c_{\mathbb{C}}$ of $P_{\mathbb{C}}$ in M, the situation is simpler.

Proposition 4.9. *Every $\mathbb{T}^d_{\mathbb{C}}$-invariant section Ψ of $L^k \to M$ vanishes over $P^c_{\mathbb{C}}$.*

Proof. This is also a consequence of Eq. (20) (cf. [13, Theorem 5.4]). □

The previous propositions were shown by Guillemin and Sternberg in [13]. Furthermore, they noticed the following important fact.

Lemma 4.10. *Let g be the Kähler metric ($g(X, Y) = \omega(X, JY)$). Then*

$$\frac{1}{2}\partial_{t_i}\partial_{t_j}\varphi(t, y) = g\big(\xi_i^\#, \xi_j^\#\big)(t, y).$$

Proof. We have $J\xi_i^\#.\langle \mu, \xi_j \rangle = \omega(J\xi_i^\#, \xi_j^\#) = -g(\xi_i^\#, \xi_j^\#)$. The result follows from Proposition 4.8. □

Then for every $y \in P$, the function $\varphi(\cdot, y)$ is strictly convex. It admits a global minimum at $t = 0$ and this minimum is $\varphi(0, y) = 0$. We obtain the following proposition.

Theorem 4.11. *Let $\epsilon > 0$, P_ϵ be the subset of $P_{\mathbb{C}}$*

$$P_\epsilon := \{(t, y) \in P_{\mathbb{C}};\ |t| < \epsilon\}$$

and P_ϵ^c its complementary subset in M. There exists some positive constants $C(\epsilon)$, C, C' such that for every k and every $\Psi \in \mathcal{H}_{\lambda,k}$,

$$(\Psi, \Psi)(x) \leqslant Ck^n e^{-kC(\epsilon)}(\Psi, \Psi)_M, \quad \forall x \in P_\epsilon^c \quad \text{and}$$
$$(\Psi, \Psi)_{P_\epsilon^c} \leqslant C'k^n e^{-kC(\epsilon)}(\Psi, \Psi)_M,$$

where $(\Psi, \Psi)_{P_\epsilon^c} := \int_{P_\epsilon^c} (\Psi, \Psi)\delta_M$.

Remark 4.12. This result shows that the eigenstates of $\mathcal{H}_{\lambda,k}$ are concentrated on P. Actually, since the $\mathrm{T}_{\xi,k}$ are Toeplitz operators with principal symbol $\langle \mu, \xi \rangle$, we could directly deduce from general properties of these operators [7] a weaker version where $Ck^n e^{-kC(\epsilon)}$ is replaced by $C_N(\epsilon)k^{-N}$ with N arbitrary large.

Proof. Let $C(\epsilon)$ be the minimum value of φ over the compact set

$$\{(t, y) \in P_{\mathbb{C}};\ |t| = \epsilon\}.$$

Since $\varphi(\cdot, y)$ is strictly convex with a global vanishing minimum at $t = 0$, $C(\epsilon)$ is positive and

$$\varphi(t, y) \geqslant C(\epsilon), \quad \forall (y, t) \in P_\epsilon^c.$$

On the other hand, using coherent states as in [6, Section 5], we prove that there exists a constant C such that for every k and $\Psi \in \mathcal{H}_k$,

$$(\Psi, \Psi)(x) \leqslant Ck^n(\Psi, \Psi)_M, \quad \forall x \in P.$$

If Ψ belongs to $\mathcal{H}_{\lambda,k}$, then it is $\mathbb{T}_{\mathbb{C}}^d$-invariant and so it satisfies Eq. (19). Furthermore, it vanishes over the complementary set of $P_{\mathbb{C}}$. This implies the first part of the result. By integrating, we get the second part with $C' = C\,\mathrm{Vol}(P_\epsilon)$. $\square$

Remark 4.13. (*Harmonic oscillator*) We have to adapt Theorem 4.11 since $\mathbb{C}^n$ is not compact. The imaginary part of the complex action is given by

$$\mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n, \quad (t, z) \to l_{it}.z = \left(e^{-t\mathfrak{p}_1} z_1, \dots, e^{-t\mathfrak{p}_n} z_n\right).$$

A section Ψ of $\mathcal{H}_{1,k}$ satisfies

$$\Psi(l_{it}.z) = e^{-kt}\Psi(z). \tag{21}$$

From this we obtain the following lemma.

Lemma 4.14. *Let $g(\cdot, k)$ be a sequence of functions on $\mathbb{C}^n$. Assume that there exist C and N such that*

$$\left|g(z,k)\right| \leqslant C\left(1+|z|\right)^N, \quad \forall z \in \mathbb{C}^n, \forall k.$$

Let $\epsilon > 0$, P_ϵ be the subset $\{l_{it}.z;\ |t| < \epsilon \text{ and } z \in P\}$ of $P_{\mathbb{C}}$ and P_ϵ^c its complementary subset in $\mathbb{C}^n$. There exist some positive constants $C(\epsilon)$, C' such that for every k and every $\Psi \in \mathcal{H}_{1,k}$,

$$\left(g(\cdot,k)\Psi,\Psi\right)_{P_\epsilon^c} \leqslant C' k^n e^{-kC(\epsilon)} (\Psi,\Psi)_{\mathbb{C}^n}.$$

Proof. Let Ψ belongs to $\mathcal{H}_{1,k}$. First, as in the compact case, there exists C_1 such that

$$\left|\Psi(z)\right|^2 e^{-k|z|^2} \leqslant C_1 k^n (\Psi,\Psi)_{\mathbb{C}^n}, \quad \forall z \in P.$$

Let $w = l_{it}.z$ with $z \in P$. It follows from (21) that

$$\left|\Psi(w)\right|^2 e^{-k|w|^2} = \left|\Psi(z)\right|^2 e^{-k|z|^2} e^{-k(|w|^2+2t-|z|^2)} \leqslant C_1 k^n e^{-k(|w|^2+2t-|z|^2)} (\Psi,\Psi)_{\mathbb{C}^n}.$$

Using that $|z|^2 \leqslant 1$, we obtain

$$\left|\Psi(w)\right|^2 e^{-k|w|^2} \leqslant C_1 k^n e^{-k} (\Psi,\Psi)_{\mathbb{C}^n} \quad \text{if } t \geqslant 1. \tag{22}$$

On the other hand, assume that $t < 0$. There exists $C_2 > 0$ such that $|z|^2 \geqslant C_2$ for every $z \in P$. So $|w|^2 = \sum e^{-2\mathfrak{p}_i t}|z_i|^2 \geqslant C_2 e^{-2t}$. Consequently

$$e^{-k(|w|^2+2t-|z|^2)} \leqslant C_3 e^{-k(|w|^2-2\ln|w|-1)} \leqslant C_3 e^{-\frac{k}{2}|w|^2} e^{-k},$$

if $|w|$ is sufficiently large. Hence there exists $t_- < 0$ such that

$$\left|\Psi(w)\right|^2 e^{-k|w|^2} \leqslant C_4 k^n e^{-k} (\Psi,\Psi)_{\mathbb{C}^n} e^{-\frac{k}{2}|w|^2} \quad \text{if } t \leqslant t_-.$$

This last equation and (22) lead to the result if $\epsilon \geqslant \max(1, -t_-)$. For the smaller values of ϵ, we complete the proof as in Theorem 4.11 since we are reduced to a compact subset of $P_{\mathbb{C}}$. □

4.5. *The integration map*

Recall that δ_M and δ_{M_r} are the Liouville measures of M and M_r, respectively, and $p_{\mathbb{C}}$ denote the projection $P_{\mathbb{C}} \to M_r$.

Let $I_k : C_o^\infty(P_{\mathbb{C}}) \to C^\infty(M_r)$ be the map given by

$$I_k(f)(x) = \left(\frac{k}{2\pi}\right)^{d/2} \left(\int_{p_{\mathbb{C}}^{-1}(x)} e^{-k\varphi} f \delta_M\right).\delta_{M_r}^{-1}(x),$$

where φ is defined in Proposition 4.8. Equivalently, $I_k(f)\delta_{M_r}$ is the push-forward of $(k/2\pi)^{\frac{d}{2}} \times e^{-k\varphi} f \delta_M$ by $p_{\mathbb{C}}$.

Remark 4.15. (*Orbifold*) If M_r is an orbifold, the push-forward by $p_{\mathbb{C}}$ of a density ν of $P_{\mathbb{C}}$ with compact support is defined as in the manifold case in such a way that:

$$\forall g \in C^{\infty}(M_r), \quad \int_{P_{\mathbb{C}}} \nu p_{\mathbb{C}}^* g = \int_{M_r} g p_{\mathbb{C}*} \nu.$$

Applying this in orbifold charts of $P_{\mathbb{C}}$ and M_r, we recover the usual definition. With the same notations as in Remark 3.8, assume that g has a compact support in $|U|$. Denote by g_U, $(p_{\mathbb{C}*}\nu)_U$ the local lifts in U of g and $p_{\mathbb{C}*}\nu$. Then by definition of an integral in an orbifold,

$$\int_{M_r} g p_{\mathbb{C}*} \nu = \frac{1}{\#G} \int_U g_U . (p_{\mathbb{C}*}\nu)_U .$$

So it follows from (11) that $(p_{\mathbb{C}*}\nu)_U$ is the push-forward of $\pi_V^* \nu$ by the projection $V \to U$. One can check that the $(p_{\mathbb{C}*}\nu)_U$ agree on overlaps and define a global section $p_{\mathbb{C}*}\nu$.

Remark 4.16. (*Harmonic oscillator*) Since we apply I_k only to functions with compact support, all the results in this section extend directly to this case.

We introduced the map I_k because it satisfies the following property.

Proposition 4.17. *For every $f \in C_o^{\infty}(P_{\mathbb{C}})$, we have*

$$(f\Psi, \Psi')_M = \big(I_k(f)\mathrm{V}_k\Psi, \mathrm{V}_k\Psi'\big)_{M_r}, \quad \forall \Psi, \Psi' \in \mathcal{H}_{\lambda,k}, \tag{23}$$

or equivalently

$$\Pi_{\lambda,k} M_f \Pi_{\lambda,k} = \mathrm{V}_k^* M_{I_k(f)} \mathrm{V}_k .$$

Proof. Let us prove (23). Since the support of f is a subset of $P_{\mathbb{C}}$,

$$(f\Psi, \Psi')_M = \int_{P_{\mathbb{C}}} f(\Psi, \Psi')\delta_M .$$

By definition of V_k, we have

$$p^*\big(\mathrm{V}_k\Psi, \mathrm{V}_k\Psi'\big) = \left(\frac{2\pi}{k}\right)^{d/2} j^*\big(\Psi, \Psi'\big).$$

So Eq. (19) can be rewritten as

$$(\Psi, \Psi') = \left(\frac{k}{2\pi}\right)^{d/2} e^{-k\varphi} p_{\mathbb{C}}^*\big(\mathrm{V}_k\Psi, \mathrm{V}_k\Psi'\big).$$

Hence,

$$\big(f\Psi,\Psi'\big)_M=\left(\frac{k}{2\pi}\right)^{d/2}\int_{P_{\mathbb{C}}} f e^{-k\varphi} p_{\mathbb{C}}^*\big(\mathrm{V}_k\Psi,\mathrm{V}_k\Psi'\big)=\int_{M_r} I_k(f)\big(\mathrm{V}_k\Psi,\mathrm{V}_k\Psi'\big)\delta_{M_r}$$
$$=\big(I_k(f)\mathrm{V}_k\Psi,\mathrm{V}_k\Psi'\big)_{M_r}.$$

which proves the result. □

For the following, we need to control the asymptotic behavior of $I_k(f)$ as k tends to infinity when f depends also on k. First observe that there is no restriction to consider only invariant functions of $C_o^\infty(P_{\mathbb{C}})$. Indeed, if $f\in C_o^\infty(P_{\mathbb{C}})$ and

$$f_\lambda(x)=\int_{\mathbb{T}^d} f(l_\theta.x)\delta_{\mathbb{T}^d}(\theta)$$

then $I_k(f)=I_k(f_\lambda)$.

Denote by $S_o(P_{\mathbb{C}})$ the set of sequences $(f(\cdot,k))_k$ of $C^\infty_{\mathbb{T}^d}(P_{\mathbb{C}})$ such that there exists a compact set $K\subset P_{\mathbb{C}}$ which contains the support of $f(\cdot,k)$ for every k and $f(\cdot,k)$ admits an asymptotic expansion for the C^∞ topology of the form

$$f(\cdot,k)=\sum_l k^{-l}f_l+O\big(k^{-\infty}\big).$$

Let us introduce a basis ξ_i of the integral lattice of $\mathfrak{t}_d$ and denote by $\xi_i^\#$ the associated vector fields of M. Recall that g is the Kähler metric. The following result involves the determinant of $g(\xi_i^\#,\xi_j^\#)$, which clearly does not depend on the choice of the basis ξ_i.

Proposition 4.18. *Let $f\in S_o(P_{\mathbb{C}})$. Then the sequence $I_k(f(\cdot,k))$ is a symbol of M_r. Furthermore*

$$p^*g_0=j^*\big(\det\big[g\big(\xi_i^\#,\xi_j^\#\big)\big]^{1/2},f_0\big),$$

where g_0 and f_0 are such that $I_k(f(\cdot,k))=g_0+O(k^{-1})$ and $f(\cdot,k)=f_0+O(k^{-1})$.

This proposition admits the following converse.

Corollary 4.19. *For every symbol $g(\cdot,k)$ of $S(M_r)$, there exists $f(\cdot,k)\in S_o(P_{\mathbb{C}})$ such that $g(\cdot,k)=I_k(f(\cdot,k))$.*

Proof. Let r be a non-negative invariant function of $C_o^\infty(P_{\mathbb{C}})$ such that $r=1$ on a neighborhood of P. Set

$$f(\cdot,k):=r.p_{\mathbb{C}}^*\big(g(\cdot,k)I_k^{-1}(r)\big).$$

From the previous proposition, $I_k(r)$ is a symbol of M_r and the first coefficient of its asymptotic expansion does not vanish. So $I_k^{-1}(r)$ is also a symbol of M_r. Consequently $f(\cdot,k)$ belongs to $S_o(P_{\mathbb{C}})$. Furthermore, we have $g(\cdot,k)=I_k(f(\cdot,k))$. □

Proof of Proposition 4.18. We integrate first over the fibers of $\mathfrak{t}_d \times P \to P$. Let us compute the Liouville measure δ_M. We denote by $t_1, \dots, t_d$ the linear coordinates of $\mathfrak{t}_d$ associated to $\xi_1, \dots, \xi_d$.

Lemma 4.20. *There exists an invariant measure δ_P on P and a function $\delta \in C^\infty(\mathfrak{t}_d \times P)$ such that*

$$\delta_M = \delta.\delta_P.|dt_1 \dots dt_d| \quad \textit{over } \mathfrak{t}_d \times P,$$
$$p_*\delta_P = \delta_{M_r} \quad \textit{and} \quad \delta(0, \cdot) = j^* \det\bigl[g\bigl(\xi_i^\#, \xi_j^\#\bigr)\bigr].$$

Proof. Let us write over $\{0\} \times P \subset \mathfrak{t}_d \times P$:

$$\omega = \beta + \sum_{1 \leqslant i \leqslant d} \beta_i \wedge dt_i + \sum_{1 \leqslant i < j \leqslant d} a_{ij}\, dt_i \wedge dt_j,$$

where $\beta \in \Omega^2(P)$, $\beta_i \in \Omega^1(P)$ and $a_{ij} \in C^\infty(P)$. Since this decomposition is unique, these forms are all invariant. Let us set

$$\delta_P = \frac{|\beta^{\wedge n_r}|}{n_r!} . \frac{|\beta_1 \wedge \dots \wedge \beta_d|}{\det[g(\xi_i^\#, \xi_j^\#)]}.$$

Since $j^*\omega = \beta$, $\beta = p^*\omega_r$. We also have $g(\xi_i^\#, \xi_j^\#) = \omega(\xi_i^\#, J\xi_j^\#) = \langle \beta_j, \xi_i^\# \rangle$. Since the ξ_i are a basis of the integer lattice, we obtain $p_*\delta_P = \delta_{M_r}$. In the case M_r is an orbifold, this can be proved using local charts of M_r and P as in Remark 4.15.

Since $\beta(\xi_i^\#, \cdot) = 0$, $(dt_i \wedge dt_j)(\xi_k^\#, \cdot) = 0$ and $(\beta_i \wedge dt_i)(\xi_k^\#, \xi_l^\#) = 0$, we have over $\{0\} \times P$

$$\omega^{\wedge n} = \frac{n!}{n_r!} \beta^{\wedge n_r} \wedge (\beta_1 \wedge dt_1) \wedge \dots \wedge (\beta_d \wedge dt_d).$$

Hence

$$\delta_M = \det\bigl[g\bigl(\xi_i^\#, \xi_j^\#\bigr)\bigr]\delta_P.|dt_1 \dots dt_d|$$

over $\{0\} \times P$, which proves the result. $\square$

Let $J_k(f)$ be the function of $C^\infty(P)$

$$J_k(f)(y) = \left(\frac{k}{2\pi}\right)^{d/2} \int_{\mathfrak{t}_d} e^{-k\varphi(t,y)} f(t, y)\delta(t, y)|dt_1 \dots dt_d|.$$

It is invariant and $p^* I_k(f) = J_k(f)$. So we just have to estimate $J_k(f)(y)$ which can be done with the stationary phase lemma. Recall that we computed in Lemma 4.10 the second derivatives of φ. The result follows. $\square$

Remark 4.21. The proof actually gives more about the map

$$F : C^\infty_{\mathbb{T}^d}(M)[\![\hbar]\!] \to C^\infty(M_r)[\![\hbar]\!], \qquad \sum \hbar^l f_l \to \sum \hbar^l g_l, \tag{24}$$

such that $I_k(f(\cdot,k)) = \sum k^{-l} g_l + O(k^{-\infty})$ if $f(\cdot,k) = \sum k^{-l} f_l + O(k^{-\infty})$.

Since F enters in the computation of the contravariant symbol of the reduced operator, let us give its properties. First it is $\mathbb{C}[\![\hbar]\!]$-linear. So

$$F = \sum \hbar^l F_l \quad \text{with } F_l : C^\infty_{\mathbb{T}^d}(M) \to C^\infty(M_r).$$

The operators F_l are of the following form:

$$p^* F_l(g) = \det{}^{1/2}\big[g\big(\xi_i^\#, \xi_j^\#\big)\big] \sum_{|\alpha| \leqslant 2l} a_{\alpha,l}\, j^*\big(\big(J\xi_1^\#\big)^{\alpha(1)} \cdots \big(J\xi_d^\#\big)^{\alpha(d)}.f\big),$$

where the functions $a_{l,\alpha}$ are polynomials in the derivatives of $\mu_i = \langle \mu, \xi_i \rangle$ and $\Delta\mu_i$ with respect to the gradient vector fields of the μ_j.

Indeed by Proposition 4.8 the derivatives of φ can be computed in terms of the derivatives of μ_i. Furthermore, it is easily proved that

$$\big(J\xi_i^\#\big).\ln\delta = \Delta\mu_i$$

with Δ the Laplace–Beltrami operator of M, which gives the derivatives of δ in terms of the derivatives of $\Delta\mu_i$. Then the computation of the functions $a_{\alpha,l}$ follows from the stationary phase lemma.

4.6. From the λ-Toeplitz operators to the reduced Toeplitz operators

We begin with a rough estimate of the maps V_k and W_k.

Proposition 4.22. *There exists a constant $C > 0$, such that for every k the uniform norms of V_k, V_k^*, W_k and W_k^* are bounded by C.*

Proof. By Corollary 4.19, there exists $f(\cdot,k) \in S_o(P_{\mathbb{C}})$ such that $I_k(f(\cdot,k)) = 1$. By Proposition 4.17,

$$\mathrm{V}_k^*\mathrm{V}_k = \Pi_{\lambda,k} M_{f(\cdot,k)} \Pi_{\lambda,k}.$$

Furthermore, it follows from Proposition 4.18 that $f(\cdot,k) = f_0 + O(k^{-1})$ with f_0 positive on P.

Since $f(\cdot,k)$ is a symbol, there exists C_1 such that $f(\cdot,k) \leqslant C_1$ over M for every k. So

$$(\mathrm{V}_k\Psi, \mathrm{V}_k\Psi)_{M_r} = \big(f(\cdot,k)\Psi, \Psi\big)_M \leqslant C_1(\Psi,\Psi)_M$$

which proves that the uniform norms of V_k and V_k^* are smaller than $C_1^{1/2}$.

Since f_0 is positive on P, there exists a neighborhood P_ϵ of P defined as in Theorem 4.11 and a constant $C_2 > 0$, such that

$$f(\cdot,k) \geqslant C_2 \quad \text{over } P_\epsilon$$

when k is sufficiently large. So

$$(\mathrm{V}_k\Psi, \mathrm{V}_k\Psi)_{M_r} = \big(f(\cdot,k)\Psi, \Psi\big)_M \geqslant \big(f(\cdot,k)\Psi, \Psi\big)_{P_\epsilon} \geqslant C_2(\Psi,\Psi)_{P_\epsilon}.$$

Furthermore, Theorem 4.11 implies that

$$(\Psi,\Psi)_{P_\epsilon} = (\Psi,\Psi)_M - (\Psi,\Psi)_{P_\epsilon^c} \geqslant \frac{1}{2}(\Psi,\Psi)_M$$

when k is sufficiently large. Consequently the uniform norms of W_k and W_k^* are smaller than $(\frac{1}{2}C_2)^{-1/2}$ when k is sufficiently large. □

Remark 4.23. (*Harmonic oscillator*) The result is still valid. There are some modifications in the proof. Instead of Theorem 4.11, we have to use Lemma 4.14. The same holds for Theorems 4.24 and 4.25.

Let us now give the relations between the λ-Toeplitz operators and the Toeplitz operators of M_r.

Theorem 4.24. *If* T_k *is a λ-Toeplitz operator, then* $\mathrm{W}_k^*\mathrm{T}_k\mathrm{W}_k$ *is a Toeplitz operator of* M_r. *Furthermore, if*

$$\mathrm{T}_k = \Pi_{\lambda,k} M_{f(\cdot,k)} \Pi_{\lambda,k} + O\big(k^{-\infty}\big),$$

with $f(\cdot,k) = \sum k^{-l} f_l + O(k^\infty)$ *an invariant symbol, then*

$$\sigma_{\mathrm{cont}}\big(\mathrm{W}_k^*\mathrm{T}_k\mathrm{W}_k\big) = F\Big(\sum \hbar^l f_l\Big),$$

where the map F *is defined in* (24). *Conversely, if* T_k *is a Toeplitz operator of* M_r, *then* $\mathrm{V}_k^*\mathrm{T}_k\mathrm{V}_k$ *is a λ-Toeplitz operator. So the map*

$$\mathcal{T}_\lambda \to \mathcal{T}_r, \quad \mathrm{T}_k \to \mathrm{W}_k^*\mathrm{T}_k\mathrm{W}_k,$$

is a bijection.

Proof. Let

$$\mathrm{T}_k = \Pi_{\lambda,k} M_{f(\cdot,k)} \Pi_{\lambda,k} + \mathrm{R}_k$$

be a λ-Toeplitz operator, where $f(\cdot,k) = \sum k^{-l} f_l + O(k^\infty)$ is an invariant symbol and R_k is $O(k^{-\infty})$. Then

$$\mathrm{W}_k^*\mathrm{T}_k\mathrm{W}_k = \mathrm{W}_k^* M_{f(\cdot,k)} \mathrm{W}_k + \mathrm{W}_k^*\mathrm{R}_k\mathrm{W}_k.$$

By Proposition 4.22, $\mathrm{W}_k^*\mathrm{R}_k\mathrm{W}_k$ is $O(k^{-\infty})$.

Let P_ϵ be a neighborhood of P defined as in Theorem 4.11. Let r be an invariant function of $C^\infty_o(P_{\mathbb{C}})$ such that $r = 1$ over P_ϵ. Write

$$\mathrm{W}_k^* M_{f(\cdot,k)} \mathrm{W}_k = \mathrm{W}_k^* M_{rf(\cdot,k)} \mathrm{W}_k + \mathrm{W}_k^* M_{(1-r)f(\cdot,k)} \mathrm{W}_k.$$

The second term on the right-hand side is $O(k^{-\infty})$. Indeed by Proposition 4.22, it suffices to prove that $\Pi_{\lambda,k} M_{(1-r)f(\cdot,k)} \Pi_{\lambda,k}$ is $O(k^{-\infty})$. We have

$$\big((1-r)f(\cdot,k)\Psi, (1-r)f(\cdot,k)\Psi\big)_M = \big((1-r)f(\cdot,k)\Psi, (1-r)f(\cdot,k)\Psi\big)_{P_\epsilon^c};$$

since $r = 1$ over P_ϵ,

$$\leqslant C(\Psi, \Psi)_{P_\epsilon^c},$$

where C does not depend of k. Theorem 4.11 leads to the conclusion.

Now $rf(\cdot,k)$ is a symbol of $S_o(P_{\mathbb{C}})$. So by Proposition 4.18, $g(\cdot,k) := I_k(rf(\cdot,k))$ is a symbol of $S(M_r)$. By Proposition 4.17,

$$\mathrm{W}_k^* M_{rf(\cdot,k)} \mathrm{W}_k = \Pi_{r,k} M_{g(\cdot,k)} \Pi_{r,k}$$

which proves that $\mathrm{W}_k^* \mathrm{T}_k \mathrm{W}_k$ is a Toeplitz operator of M_r with contravariant symbol $F(\sum \hbar^l f_l)$.

Conversely, let

$$\mathrm{T}_k = \Pi_{r,k} M_{g(\cdot,k)} \Pi_{r,k} + \mathrm{R}_k$$

be a Toeplitz operator of M_r, where $g(\cdot,k) \in S(M_r)$ and R_k is $O(k^{-\infty})$. Write

$$\mathrm{V}_k^* \mathrm{T}_k \mathrm{V}_k = \mathrm{V}_k^* M_{g(\cdot,k)} \mathrm{V}_k + \mathrm{V}_k^* \mathrm{R}_k \mathrm{V}_k.$$

Then, $\mathrm{V}_k^* \mathrm{R}_k \mathrm{V}_k$ is $O(k^{-\infty})$ by Proposition 4.22. By Corollary 4.19, there exists $f(\cdot,k) \in S_o(P_{\mathbb{C}})$ such that

$$\mathrm{V}_k^* M_{g(\cdot,k)} \mathrm{V}_k = \Pi_{\lambda,k} M_{f(\cdot,k)} \Pi_{\lambda,k}.$$

Consequently $\mathrm{V}_k^* \mathrm{T}_k \mathrm{V}_k$ is a λ-Toeplitz operator. □

Recall that $\mathrm{U}_k = \mathrm{V}_k (\mathrm{V}_k^* \mathrm{V}_k)^{-1/2} : L^2(M, L^k) \to L^2(M_r, L_r^k)$. Let us state our main result.

Theorem 4.25. *The map*

$$\mathcal{T}_\lambda \to \mathcal{T}_r, \quad \mathrm{T}_k \to \mathrm{U}_k \mathrm{T}_k \mathrm{U}_k^*,$$

is an isomorphism of $*$-algebra. Furthermore, if

$$\mathrm{T}_k = \Pi_{\lambda,k} M_{f(\cdot,k)} \Pi_{\lambda,k} + O\big(k^{-\infty}\big),$$

with $f(\cdot,k)=\sum k^{-l} f_l + O(k^{\infty})$ *an invariant symbol, then*

$$\sigma_{\mathrm{cont}}\big(\mathrm{U}_k\mathrm{T}_k\mathrm{U}_k^*\big)=e^{-1/2}*_{\mathrm{cr}} F\Big(\sum \hbar^l f_l\Big)*_{\mathrm{cr}} e^{-1/2},$$

where $e=F(1)$ *and* $e^{-1/2}$ *is the formal series of* $C^{\infty}(M_r)[\![\hbar]\!]$ *whose first coefficient is positive and such that* $e^{-1/2}*_{\mathrm{cr}} e^{-1/2}*_{\mathrm{cr}} e=1$.

Again, in the proof we use some basic properties of the Toeplitz operators of M_r, which are known in the manifold case and will be extended to the orbifold case in Section 6.

Proof. It is easily checked that

$$\mathrm{U}_k=\big(\mathrm{W}_k^*\mathrm{W}_k\big)^{-1/2}\mathrm{W}_k^*.$$

Let T_k be a λ-Toeplitz operator. We have

$$\mathrm{U}_k\mathrm{T}_k\mathrm{U}_k^*=\big(\mathrm{W}_k^*\mathrm{W}_k\big)^{-1/2}\mathrm{W}_k^*\mathrm{T}_k\mathrm{W}_k\big(\mathrm{W}_k^*\mathrm{W}_k\big)^{-1/2}. \tag{25}$$

By Theorem 4.24, $\mathrm{W}_k^*\mathrm{W}_k$ is a Toeplitz operator of M_r with a positive principal symbol. It follows from the functional calculus for Toeplitz operator (cf. [6]) that $(\mathrm{W}_k^*\mathrm{W}_k)^{-1/2}$ is a Toeplitz operator also. Now $\mathrm{W}_k^*\mathrm{T}_k\mathrm{W}_k$ is a Toeplitz operator by Theorem 4.24. Since the Toeplitz operators of M_r form an algebra, $\mathrm{U}_k\mathrm{T}_k\mathrm{U}_k^*$ is a Toeplitz operator. The computation of its covariant symbol is also a consequence of (25). Indeed by Theorem 4.24, the symbol of $\mathrm{W}_k^*\mathrm{T}_k\mathrm{W}_k$ and $\mathrm{W}_k^*\mathrm{W}_k$ are $F(\sum \hbar^l f_l)$ and e, respectively.

Conversely, if S_k is a Toeplitz operator of M_r, then

$$\begin{aligned}
\mathrm{U}_k^*\mathrm{S}_k\mathrm{U}_k&=\mathrm{W}_k\big(\mathrm{W}_k^*\mathrm{W}_k\big)^{-1/2}\mathrm{S}_k\big(\mathrm{W}_k^*\mathrm{W}_k\big)^{-1/2}\mathrm{W}_k^*\\
&=\big(\mathrm{V}_k^*\mathrm{W}_k^*\big)\mathrm{W}_k\big(\mathrm{W}_k^*\mathrm{W}_k\big)^{-1/2}\mathrm{S}_k\big(\mathrm{W}_k^*\mathrm{W}_k\big)^{-1/2}\mathrm{W}_k^*(\mathrm{W}_k\mathrm{V}_k)\\
&=\mathrm{V}_k^*\big(\mathrm{W}_k^*\mathrm{W}_k\big)^{1/2}\mathrm{S}_k\big(\mathrm{W}_k^*\mathrm{W}_k\big)^{1/2}\mathrm{V}_k.
\end{aligned}$$

And in a similar way, we deduce from theorem 4.24 that $\mathrm{U}_k^*\mathrm{S}_k\mathrm{U}_k$ is a λ-Toeplitz operator. □

Remark 4.26. (*Symbolic calculus*) Recall that we computed the operator F at the end of Section 4.5. Furthermore, the star-product $*_{\mathrm{cr}}$ can be computed in terms of the Kähler metric of M_r (cf. [6]). This leads to the computation of the contravariant symbol of the reduced operator $\mathrm{U}_k\mathrm{T}_k\mathrm{U}_k^*$ in terms of the multiplicator $\sum k^{-l} f_l$ defining the λ-Toeplitz operator T_k. In particular, the principal symbol g_0 of $\mathrm{U}_k\mathrm{T}_k\mathrm{U}_k^*$ is such that $p^*g_0=i^*f_0$.

Remark 4.27. (*Proof of Theorem 4.6*) Because of the previous remark, the λ-Toeplitz operator T_k and the Toeplitz operator $\mathrm{U}_k\mathrm{T}_k\mathrm{U}_k^*$ have the same principal symbol. Consequently all the assertions of Theorem 4.6 follow from Theorem 4.25 and the calculus of the contravariant symbol for the Toeplitz operators of M_r.

Remark 4.28. (*Proof of Theorem 4.4*) Theorem 4.4 stated in the introduction is a consequence of Theorems 4.5 and 4.25. To compute the contravariant symbol of the reduced operator, we have first to average the contravariant symbol of the Toeplitz operator of M and then apply the formula of Theorem 4.25.

Remark 4.29. (V_k *is not unitary*) The fact that the Guillemin–Sternberg isomorphism is not unitary, even after the rescaling with the factor $(\frac{k}{2\pi})^{d/4}$, can be deduced from the spectral properties of the Toeplitz operators. Indeed by Theorem 4.24, $\mathrm{W}_k^*\mathrm{W}_k$ is a Toeplitz operator with principal symbol g_0 such that

$$p^* g_0 = j^* \det\bigl[g\bigl(\xi_i^\#, \xi_j^\#\bigr)\bigr]^{1/2}.$$

Denote by m and M the minimum and maximum of g_0. Then the smallest eigenvalue E_s of $\mathrm{W}^*\mathrm{W}$ and the biggest E_S are estimated by

$$E_s = m + O\bigl(k^{-1}\bigr), \qquad E_S = M + O\bigl(k^{-1}\bigr).$$

So when the function $\det[g(\xi_i^\#, \xi_j^\#)]^{1/2}$ is not constant over P, W_k is not unitary when k is sufficiently large. This happens for instance in the case of the harmonic oscillator when the reduced space is not a manifold.

Remark 4.30. From a semi-classical point of view, the operator U_k is not unique. Indeed we can replace it with any operator of the form

$$\mathrm{T}_k \mathrm{U}_k \mathrm{S}_k$$

with S_k a unitary λ-Toeplitz operator and T_k a unitary Toeplitz operator of M_r. We can state a theorem similar to Theorem 4.25 with this operator. The only changes are in the symbolic calculus.

5. Fourier integral operators

In this section, we prove that the λ-Toeplitz operators, the Guillemin–Sternberg isomorphism and its unitarization are Fourier integral operators in the sense of [7]. Using this we can interpret the relations between these operators and the Toeplitz operators as compositions of Fourier integral operators corresponding to compositions of canonical relations.

We assume that the reduced space M_r is a manifold. A part of the material will be adapted to orbifolds in Section 7.

5.1. Definitions

We first recall some definitions of [7]. Let M_1 and M_2 be compact Kähler manifolds endowed with prequantization bundles $L_1^\kappa \to M_1$ and $L_2^\kappa \to M_2$. Here κ is some fixed positive integer and in the following k is always a positive multiple of k. Denote by $\mathcal{H}_k^1$ (respectively $\mathcal{H}_k^2$) the space of holomorphic sections of L_1^k (respectively L_2^k) and by Π_k^1 (respectively Π_k^2) the orthogonal projector onto $\mathcal{H}_k^1$ (respectively $\mathcal{H}_k^2$).

Consider a sequence $(\mathrm{T}_k)_{k\in\kappa\mathbb{N}^*}$ such that for every k, T_k is an operator $\mathcal{H}_k^2 \to \mathcal{H}_k^1$. As previously we extend T_k to the Hilbert space of sections with finite norm in such a way that it vanishes on the orthogonal of $\mathcal{H}_k^2$. The Schwartz kernel T_k is the section of $L_1^k \boxtimes L_2^{-k} \to M_1 \times M_2$ such that

$$\mathrm{T}_k.\Psi(x_1) = \int_{M_2} T_k(x_1,x_2).\Psi(x_2)\delta_{M_2}(x_2),$$

where δ_{M_2} is the Liouville measure of M_2. All the operators we consider in this section are of this form.

5.1.1. Smoothing operators

A sequence $(f(\cdot,k))$ of functions on a manifold X is $O_\infty(k^{-\infty})$ if for every compact set K, every $N \geqslant 0$, every vector fields $Y_1,\dots,Y_N$ on X and every l, there exists C such that

$$\left|Y_1.Y_2\dots Y_N f(\cdot,k)\right| \leqslant Ck^{-l} \quad \text{on } K.$$

Let $L_X \to X$ be a Hermitian line bundle. Let (Ψ_k) be a sequence such that for every k, Ψ_k is a section of L_X^k. Then (Ψ_k) is $O_\infty(k^{-\infty})$ if for every local unitary section $t : V \to L_X$, the sequence $(f(\cdot,k))$ such that $\Psi_k = f(\cdot,k)t^k$ is $O_\infty(k^{-\infty})$.

We say that an operator (T_k) is smoothing if the sequence (T_k) of Schwartz kernels is $O_\infty(k^{-\infty})$. Clearly if T_k is $O_\infty(k^{-\infty})$, the compacity of $M_1 \times M_2$ implies that the uniform norm of T_k is $O(k^{-\infty})$. If $\Pi_k^1 \mathrm{T}_k \Pi_k^2 = \mathrm{T}_k$ for every k, then the converse is true.

5.1.2. Fourier integral operators

If ω_2 is the symplectic form of M_2, we denote by M_2^- the manifold M_2 endowed with the symplectic form $-\omega_2$. The data to define a Fourier integral operator are a Lagrangian submanifold Γ of $M_1 \times M_2^-$, a flat unitary section t_Γ^κ of $L_1^\kappa \boxtimes L_2^{-\kappa} \to \Gamma$ and a formal series $\sum \hbar^l g_l$ of $C^\infty(\Gamma)[\![\hbar]\!]$.

By definition T_k is a *Fourier integral operator* associated to $(\Gamma, t_\Gamma^\kappa)$ with *total symbol* $\sum \hbar^l g_l$ if on every compact set $K \subset M_1 \times M_2$ such that $K \cap \Gamma = \emptyset$,

$$T_k(x_1,x_2) = O_\infty\big(k^{-\infty}\big).$$

Furthermore, on a neighborhood U of Γ,

$$T_k(x_1,x_2) = \left(\frac{k}{2\pi}\right)^{n(\Gamma)} E_\Gamma^k(x_1,x_2) f(x_1,x_2,k) + O_\infty\big(k^{-\infty}\big), \tag{26}$$

where

(i) E_Γ^κ is a section of $L_1^\kappa \boxtimes L_2^{-\kappa} \to U$ such that $E_\Gamma^\kappa = t_\Gamma^\kappa$ over Γ, and for every holomorphic vector field Z_1 of M_1 and Z_2 of M_2

$$\nabla_{(Z_1,0)} E_\Gamma^\kappa \equiv 0 \quad \text{and} \quad \nabla_{(0,\bar{Z}_2)} E_\Gamma^\kappa \equiv 0$$

modulo a section which vanishes to any order along Γ. Furthermore,

$$\left|E^{\kappa}_{\Gamma}(x_1, x_2)\right| < 1$$

if $(x_1, x_2) \notin \Gamma$;

(ii) $(f(\cdot, k))_k$ is a symbol of $S(U)$ with an asymptotic expansion $\sum k^{-l} f_l$ such that

$$f_l = g_l \quad \text{over } \Gamma$$

and $(Z_1, 0).f_l \equiv 0$ and $(0, \bar{Z}_2).f_l \equiv 0$ modulo a function which vanishes to any order along Γ for every holomorphic vector fields Z_1 of M_1 and Z_2 of M_2;

$n(\Gamma)$ is a real number. Denote by $\mathcal{F}(\Gamma, t^{\kappa}_{\Gamma})$ the set of Fourier integral operators associated to $(\Gamma, t^{\kappa}_{\Gamma})$.

Theorem 5.1. *The map $\mathcal{F}(\Gamma, t^{\kappa}_{\Gamma}) \to C^{\infty}(\Gamma)[\![\hbar]\!]$ which sends an operator into its total symbol is well defined and onto. Its kernel consists of the operators $O(k^{-\infty})$.*

The *principal symbol* of $\mathrm{T}_k \in \mathcal{F}(\Gamma, t^{\kappa}_{\Gamma})$ is the first coefficient $g_0 \in C^{\infty}(\Gamma)$ of the total symbol. If it does not vanish, T_k is said *elliptic*. In [7], we proved the basic results regarding the composition properties of this type of Fourier integral operators.

5.2. Toeplitz operators

The first example of Fourier integral operators are the Toeplitz operators. The diagonal Δ_r is a Lagrangian submanifold of $M_r \times M_r^-$. Denote by $t^{\kappa}_{\Delta_r}$ the flat section of $L^{\kappa}_r \boxtimes L^{-\kappa}_r \to \Delta_r$ such that

$$t^{\kappa}_{\Delta_r}(x, x) = z \otimes z^{-1} \quad \text{if } z \in L^{\kappa}_x \text{ and } z \neq 0.$$

Definition 5.2. $\mathcal{F}_r$ is the space of Fourier integral operators associated to $(\Delta_r, t^{\kappa}_{\Delta_r})$ with $n(\Delta_r) = n_r$ the complex dimension of M_r.

By identifying Δ_r with M_r, we consider the total symbols of these operators as formal series of $C^{\infty}(M_r)[\![\hbar]\!]$. Our main result in [6] was the following theorem.

Theorem 5.3. *Every Toeplitz operator (T_k) of M_r is a Fourier integral operator associated to $(\Delta_r, t^{\kappa}_{\Delta_r})$ and conversely. Furthermore, there exists an equivalence of star-products*

$$E : C^{\infty}(M_r)[\![\hbar]\!] \to C^{\infty}(M_r)[\![\hbar]\!]$$

such that if (T_k) is a the Toeplitz operator with contravariant symbol $\sum \hbar^l f_l$, then the total symbol of (T_k) as a Fourier integral operator is $E(\sum \hbar^l f_l)$.

The same result holds for the Toeplitz operators of M. In the following we use the notations Δ, t_Δ and $\mathcal{F}$ corresponding to Δ_r, $t^{\kappa}_{\Delta_r}$ and $\mathcal{F}_r$ on M.

5.3. The λ-Toeplitz operators

Recall some notations of Section 3. The action of $\theta \in \mathbb{T}^d$ on M (respectively L^k) is denoted by l_θ (respectively $\mathcal{L}_\theta$). P is the level set $\mu^{-1}(\lambda)$ and p is the projection $P \to M_r$.

Let Λ be the moment Lagrangian

$$\Lambda = \left\{(l_\theta.x, x);\ x \in P \text{ and } \theta \in \mathbb{T}^d\right\}$$

introduced by Weinstein in [25]. Λ is a Lagrangian manifold of $M \times M^-$. Let t^κ_Λ be the section of $L^\kappa \boxtimes L^{-\kappa} \to \Lambda$ such that

$$t^\kappa_\Lambda(l_\theta.x, x) = \mathcal{L}_\theta.z \otimes z^{-1} \quad \text{if } z \in L^\kappa_x \text{ and } z \neq 0.$$

This is a flat section with constant norm equal to 1.

Definition 5.4. $\mathcal{F}_\lambda$ is the set of Fourier integral operators T_k associated to Λ and t^κ_Λ with $n(\Lambda) = n - d/2$ and such that

$$\mathcal{L}^*_\theta \mathrm{T}_k = \mathrm{T}_k \mathcal{L}^*_\theta = \mathrm{T}_k, \quad \forall\theta$$

or equivalently $\Pi_{\lambda,k}\mathrm{T}_k\Pi_{\lambda,k} = \mathrm{T}_k$.

We will deduce the following result from the fact that the algebra $\mathcal{T}_\lambda$ is isomorphic to the algebra $\mathcal{T}_r$ of Toeplitz operators of M_r (Theorems 4.24 and 4.25).

Theorem 5.5. *$\mathcal{F}_\lambda$ is the algebra $\mathcal{T}_\lambda$ of λ-Toeplitz operators.*

A similar characterization was given by Guillemin–Sternberg for pseudodifferential operators in [14]. Their proof starts from the fact that $\Pi_{\lambda,k}$ is a Fourier integral operator of $\mathcal{F}_\lambda$. If T_k is a Toeplitz operator of M, then the symbolic calculus of Fourier integral operators implies that $\Pi_{\lambda,k}\mathrm{T}_k\Pi_{\lambda,k}$ belongs to $\mathcal{F}_\lambda$. This follows essentially from the composition of canonical relations

$$\Lambda \circ \Delta \circ \Lambda = \Lambda.$$

In the same way, we can show that $\mathcal{F}_\lambda$ is a $*$-algebra (cf. Theorem 4.5), define the principal symbol and prove Theorem 18. The corresponding compositions of canonical relations are

$$\Lambda \circ \Lambda = \Lambda, \quad \Lambda^t = \Lambda.$$

The difficulty of this approach is that it uses the properties of composition of the Fourier integral operators. In addition, the composition of Λ with itself is not transverse. But it has the advantage to be more general and can be transposed in other contexts.

5.4. The Guillemin–Sternberg isomorphism and its unitarization

Let Θ be the Lagrangian submanifold of $M_r \times M^-$

$$\Theta = \{(p(x), x);\ x \in P\}.$$

Recall that $L_r^\kappa \to M_r$ is the quotient of $L^\kappa \to P$ by the action of $\mathbb{T}^d$. If $z \in L_x^\kappa$ where $x \in P$, we denote by $[z] \in L^\kappa_{r,p(x)}$ its equivalence class. Then define the section t_Θ^κ of $L_r^\kappa \boxtimes L^{-\kappa} \to \Theta$ by

$$t_\Theta^\kappa(p(x), x) = [z] \otimes z^{-1} \quad \text{if } z \in L_x^\kappa \text{ and } z \neq 0.$$

It is flat with constant norm equal to 1.

Definition 5.6. $\mathcal{F}_{\lambda,r}$ is the space of Fourier integral operators T_k associated to $(\Theta, t_\Theta^\kappa)$ with $n(\Theta) = n - 3d/4$ and such that

$$\mathrm{T}_k \mathcal{L}_\theta^* = \mathrm{T}_k, \quad \forall \theta$$

or equivalently $\mathrm{T}_k \Pi_{\lambda,k} = \mathrm{T}_k$.

Theorem 5.7. *The Guillemin–Sternberg isomorphism* V_k *and the unitary operator* $\mathrm{U}_k = \mathrm{V}_k(\mathrm{V}_k^*\mathrm{V}_k)^{-1/2}$ *are elliptic operators of* $\mathcal{F}_{\lambda,r}$.

This result is coherent with the fact that U_k induces an isomorphism between the algebra of the λ-Toeplitz operators and the algebra of the Toeplitz operators of M_r. Indeed, observe that

$$\Theta \circ \Lambda \circ \Theta^t = \Delta_r, \qquad \Theta^t \circ \Delta_r \circ \Theta = \Lambda$$

which corresponds to the equalities

$$\mathrm{U}_k \mathrm{T}_k \mathrm{U}_k^* = \mathrm{S}_k, \qquad \mathrm{U}_k^* \mathrm{S}_k \mathrm{U}_k = \mathrm{T}_k,$$

where T_k is a λ-Toeplitz operator and S_k the reduced Toeplitz operator.

To prove Theorems 5.5 and 5.7, we first explain how the spaces $\mathcal{F}_{\lambda,r}$, $\mathcal{F}_\lambda$, $\mathcal{F}_r$ of Fourier integral operators are related by the Guillemin–Sternberg isomorphism. Then we deduce Theorems 5.5 and 5.7 from Theorem 4.24.

5.5. The relations between $\mathcal{F}_\lambda$, $\mathcal{F}_{\lambda,r}$ and $\mathcal{F}_r$

Since we consider Fourier integral operators which are equivariant, we need an equivariant version of Theorem 5.1. Let us consider the same data as in Section 5.1. Let G be a compact Lie group which acts on $M_1 \times M_2$ preserving the Kähler structure of $M_1 \times M_2^-$. Assume that this action lifts to $L_1^\kappa \boxtimes L_2^{-\kappa}$, preserving the Hermitian structure and connection.

Suppose that the Lagrangian manifold Γ and the section t_Γ^κ is G-invariant. Let us denote by $\mathcal{F}_G(\Gamma, t_\Gamma^\kappa)$ the space of Fourier integral operators associated to $(\Gamma, t_\Gamma^\kappa)$ whose kernel is G-invariant.

Theorem 5.8. *The total symbol of an operator* $\mathrm{T}_k \in \mathcal{F}_G(\Gamma, t_\Gamma^\kappa)$ *is* G*-invariant. Furthermore, the total symbol map* $\mathcal{F}_G(\Gamma, t_\Gamma^\kappa) \to C_G^\infty(\Gamma)[\![\hbar]\!]$ *is onto.*

Proof. Let $\mathrm{T}_k \in \mathcal{F}_G(\Gamma, t_\Gamma^\kappa)$. Its kernel is of the form (26) on a neighborhood of Γ. Hence,

$$T_k(x_1, x_2) = \left(\frac{k}{2\pi}\right)^{n(\Gamma)} f(x_1, x_2, k) t_\Gamma^k(x_1, x_2) + O\left(k^{-\infty}\right)$$

for every $(x_1, x_2) \in \Gamma$. It follows that the total symbol is G-invariant. Conversely, if the total symbol is G-invariant, we can define a G-invariant kernel of the form (26) with a G-invariant neighborhood U, a G-invariant section E_Γ^κ and a G-invariant sequence $f(\cdot, k)$. The operator obtained T'_k does not necessarily satisfy

$$\Pi_k^1 \mathrm{T}'_k \Pi_k^2 = \mathrm{T}'_k.$$

So we set

$$\mathrm{T}_k = \Pi_k^1 \mathrm{T}'_k \Pi_k^2.$$

Using that Π_k^1 and Π_k^2 are Fourier integral operators associated to the diagonal of M_1 and M_2 respectively, we prove that the kernels of T_k and T'_k are the same modulo $O_\infty(k^{-\infty})$. Consequently T_k belongs to $\mathcal{F}_G(\Gamma, t_\Gamma^\kappa)$ and has the required symbol. □

Corollary 5.9. *There is a natural identification between total symbols of operators of* $\mathcal{F}_\lambda$ *(respectively* $\mathcal{F}_{\lambda,r}$*) and formal series of* $C^\infty(M_r)[\![\hbar]\!]$.

Proof. By Theorem 5.8, the total symbols of the operators of $\mathcal{F}_\lambda$ are the formal series of $C^\infty(\Lambda)[\![\hbar]\!]$ invariant with respect to the action of $\mathbb{T}^d \times \mathbb{T}^d$ on $\Lambda \subset M^2$. The map

$$\Lambda \to M_r, \quad (y, x) \to p(x),$$

is a $(\mathbb{T}^d \times \mathbb{T}^d)$-principal bundle. So there is a one-to-one correspondence between $C^\infty_{\mathbb{T}^d \times \mathbb{T}^d}(\Lambda)$ and $C^\infty(M_r)$. The proof is the same for the total symbols of the operators of $\mathcal{F}_{\lambda,r}$. □

Theorem 5.10. *The following maps*

$$\mathcal{F}_\lambda \to \mathcal{F}_{\lambda,r}, \quad \mathrm{T}_k \to \mathrm{V}_k \mathrm{T}_k,$$
$$\mathcal{F}_\lambda \to \mathcal{F}_r, \quad \mathrm{T}_k \to \mathrm{V}_k \mathrm{T}_k \mathrm{V}_k^*,$$

are well defined and bijective. Furthermore, if $\mathrm{T}_k \in \mathcal{F}_\lambda$*, the total symbols of* $\mathrm{V}_k\mathrm{T}_k$*,* $\mathrm{V}_k\mathrm{T}_k\mathrm{V}_k^*$ *and* T_k *are the same with the identifications of Corollary* 5.9.

Proof. These properties follows immediately from the definition of the Fourier integral operators. Indeed consider two operators

$$\mathrm{T}_k : \mathcal{H}_{\lambda,k} \to \mathcal{H}_{\lambda,k}, \qquad \mathrm{S}_k : \mathcal{H}_{\lambda,k} \to \mathcal{H}_{r,k}.$$

Extend them to the space of L^2 sections in such a way that they vanish on the orthogonal of $\mathcal{H}_{\lambda,k}$. Then $\mathrm{S}_k = \mathrm{V}_k \mathrm{T}_k$ or equivalently $\mathrm{T}_k = \mathrm{W}_k \mathrm{S}_k$ if and only if the kernels of T_k and S_k satisfy

$$T_k = \left(\frac{k}{2\pi}\right)^{d/4} (p_{\mathbb{C}} \boxtimes \mathrm{Id})^* S_k \quad \text{over } P_{\mathbb{C}} \times M. \tag{27}$$

This follows from the definition of V_k (cf. Definition 3.7). Furthermore, by Proposition 4.9, the kernel of T_k vanishes over $P_{\mathbb{C}}^c \times M$. So we may recover the kernel of T_k from the kernel of S_k and conversely. Using this we can directly check that $\mathrm{T}_k \in \mathcal{F}_\lambda$ if and only if $\mathrm{S}_k \in \mathcal{F}_{\lambda,r}$.

To do this observe that the data which define the operators of $\mathcal{F}_\lambda$ and $\mathcal{F}_{\lambda,r}$ are related in the following way:

$$(p \times \mathrm{Id})^{-1}(\Theta) = \Lambda, \qquad (p \boxtimes \mathrm{Id})^* t^\kappa_\Theta = t^\kappa_\Lambda.$$

Using that $p_{\mathbb{C}} : P_{\mathbb{C}} \to M_r$ is a holomorphic map, it comes that

$$(p_{\mathbb{C}} \boxtimes \mathrm{Id})^* E^\kappa_\Theta = E^\kappa_\Lambda.$$

That $\mathrm{T}_k \to \mathrm{V}_k \mathrm{T}_k$ is well defined, bijective and preserves the total symbols follows easily.

For the second map, we can proceed in a similar way. Let S_k be such that $\Pi_{r,k} \mathrm{S}_k \Pi_{r,k} = \mathrm{S}_k$. Then computing successively the Schwartz kernels of $\mathrm{W}_k \mathrm{S}_k$, $(\mathrm{W}_k \mathrm{S}_k)^*$, $\mathrm{W}_k (\mathrm{W}_k \mathrm{S}_k)^*$ and using

$$\bigl(\mathrm{W}_k (\mathrm{W}_k \mathrm{S}_k)^*\bigr)^* = \mathrm{W}_k \mathrm{S}_\mathrm{k} \mathrm{W}_k^*,$$

we obtain that the kernels of $\mathrm{T}_k = \mathrm{W}_k \mathrm{S}_\mathrm{k} \mathrm{W}_k^*$ and S_k satisfy

$$T_k(x,y) = \left(\frac{k}{2\pi}\right)^{d/2} e^{-k\varphi(y)} (p_{\mathbb{C}} \boxtimes p_{\mathbb{C}})^* S_k(x,y) \quad \text{over } P_{\mathbb{C}} \times P_{\mathbb{C}},$$

where the function φ has been defined in Proposition 4.8. $\square$

5.6. *Proofs of Theorems 5.5 and 5.7*

Recall that by Theorem 4.24, there is a bijection from $\mathcal{T}_\lambda$ onto $\mathcal{T}_r$

$$\mathcal{T}_\lambda \to \mathcal{T}_r, \quad \mathrm{T}_k \to \mathrm{W}_k^* \mathrm{T}_k \mathrm{W}_k. \tag{28}$$

Furthermore, we know that $\mathcal{T}_r = \mathcal{F}_r$ by Theorem 5.3. In Theorem 5.10, we proved that the map

$$\mathcal{F}_\lambda \to \mathcal{F}_r, \quad \mathrm{T}_k \to \mathrm{V}_k \mathrm{T}_k \mathrm{V}_k^*, \tag{29}$$

is a bijection. Let us deduce that $\mathcal{T}_\lambda = \mathcal{F}_\lambda$.

Let T_k belong to $\mathcal{F}_\lambda$. By (29), $\mathrm{V}_k \mathrm{T}_k \mathrm{V}_k^*$ belongs to $\mathcal{T}_r$. By (28), $\mathrm{W}_k^* \mathrm{W}_k$ belongs to $\mathcal{T}_r$. Since the product of Toeplitz operators is a Toeplitz operator,

$$\bigl(\mathrm{W}_k^* \mathrm{W}_k\bigr)\bigl(\mathrm{V}_k \mathrm{T}_k \mathrm{V}_k^*\bigr)\bigl(\mathrm{W}_k^* \mathrm{W}_k\bigr) = \mathrm{W}_k^* \mathrm{T}_k \mathrm{W}_k$$

belongs to $\mathcal{T}_r$. By (28) T_k belongs to $\mathcal{T}_\lambda$.

Conversely assume that T_k belongs to $\mathcal{T}_\lambda$. By (28), $\mathrm{W}_k^*\mathrm{T}_k\mathrm{W}_k$ belongs to $\mathcal{T}_r$. Write

$$\mathrm{V}_k\mathrm{T}_k\mathrm{V}_k^* = \left(\mathrm{V}_k\mathrm{V}_k^*\right)\left(\mathrm{W}_k^*\mathrm{T}_k\mathrm{W}_k\right)\left(\mathrm{V}_\mathrm{k}\mathrm{V}_k^*\right).$$

Observe that $\mathrm{V}_k\mathrm{V}_k^*$ is the inverse of $\mathrm{W}_k^*\mathrm{W}_k$ in the sense of Toeplitz operators, i.e.

$$\Pi_{r,k}\left(\mathrm{V}_k\mathrm{V}_k^*\right)\Pi_{r,k} = \mathrm{V}_k\mathrm{V}_k^*,$$
$$\left(\mathrm{V}_k\mathrm{V}_k^*\right)\left(\mathrm{W}_k^*\mathrm{W}_k\right) = \left(\mathrm{W}_k^*\mathrm{W}_k\right)\left(\mathrm{V}_k\mathrm{V}_k^*\right) = \Pi_{r,k}.$$

Since $(\mathrm{W}_k^*\mathrm{W}_k)$ is a Toeplitz operator with a non-vanishing symbol by Theorem 4.24, $\mathrm{V}_k\mathrm{V}_k^*$ is a Toeplitz operator. Consequently $\mathrm{V}_k\mathrm{T}_k\mathrm{V}_k^*$ belongs to $\mathcal{T}_r$ and by (29), T_k belongs to $\mathcal{F}_\lambda$.

Let us prove Theorem 5.7. By Theorem 5.5, $\Pi_{\lambda,k}$ belongs to $\mathcal{F}_\lambda$. So Theorem 5.10 implies that $\mathrm{V}_k = \mathrm{V}_k\Pi_{\lambda,k}$ belongs to $\mathcal{F}_{\lambda,r}$. Let us consider now U_k. We have

$$\mathrm{U}_k = \left(\mathrm{W}_k^*\mathrm{W}_k\right)^{-1/2}\mathrm{W}_k^* = \mathrm{V}_k\left(\mathrm{W}_k\left(\mathrm{W}_k^*\mathrm{W}_k\right)^{-1/2}\mathrm{W}_k^*\right).$$

As we saw in the proof of Theorem 4.25, $(\mathrm{W}_k^*\mathrm{W}_k)^{-1/2}$ belongs to $\mathcal{T}_r$. So by Theorem 4.24, $\mathrm{W}_k(\mathrm{W}_k^*\mathrm{W}_k)^{-1/2}\mathrm{W}_k^*$ belongs to $\mathcal{T}_\lambda = \mathcal{F}_\lambda$. And Theorem 5.10 implies that U_k belongs to $\mathcal{F}_{\lambda,r}$.

6. Toeplitz operators on orbifold

In this part, we prove the basic results about the Toeplitz operators on the orbifold M_r. We describe their kernels as Fourier integral operators associated to the diagonal, prove that the set of Toeplitz operators is an algebra and describe the associated symbolic calculus. Finally we compute the asymptotic of the density of states of a Toeplitz operator.

6.1. Schwartz kernel on orbifold

Let us introduce some notations and state some basic facts about kernels of operators on orbifolds. Let X be a reduced orbifold with a vector bundle $E \to X$. So every chart $(|U|, U, G, \pi_U)$ of X is endowed with a G-bundle $E_U \to U$. G acts effectively on U. If $g \in G$, we denote by $a_g : U \to U$ its action on U and by $\mathcal{A}_g : E_U \to E_U$ its lift. If s is a section of $E \to X$, we denote by s_U the corresponding invariant section of E_U.

As in the manifold case, we can define the dual bundle of E, the tensor product of two bundles over X, the orbifold X^2 and the bundle $E \boxtimes E^* \to X^2$. Let δ be a volume form of X. Then every section T of $E \boxtimes E^*$ defines an operator T in the following way. Consider two charts $(|U|, U, G, \pi_U)$ and $(|V|, V, H, \pi_V)$ of X. Then T is given over $|U| \times |V|$ by a $(G \times H)$-invariant section T_{UV} of $E_U \boxtimes E_V^*$. If s is a section of E with compact support in $|V|$, then

$$(\mathrm{T}s)_U = \frac{1}{\#H}\int_V T_{UV}.s_V\delta_V.$$

Assume that E is Hermitian and define the scalar product of sections of E by using δ. If T is an operator which acts on $C^\infty(X, E)$, vanishing over the orthogonal of a finite-dimensional subspace of $C^\infty(X, E)$, it is easily proved that T has a Schwartz kernel T. It is unique. Furthermore, if E has rank one, the trace of T is given by

$$\mathrm{Tr}(\mathrm{T}) = \int_X \Delta^* T \delta,$$

where $\Delta : X \to X^2$ is the diagonal map. Since Δ is a good map (in the sense of [8]), the pull-back $\Delta^*(E \boxtimes E^*)$ is well defined. It is naturally isomorphic to $E \otimes E^* \to X$. So $\Delta^* T$ is a section of $E \otimes E^* \simeq \mathbb{C}$. Finally the previous integral is defined with the orbifold convention: if $(|U|, U, G, \pi_U)$ is a chart of X and $\Delta^* T$ has support in $|U|$, it is given by

$$\frac{1}{\# G} \int_U (\Delta^* T)_U \delta_U.$$

Note also that it is false that every section of $E \otimes E^*$ is the pull-back by Δ of a section of $E \boxtimes E^*$. For instance when E is a line bundle, $E \otimes E^* \simeq \mathbb{C}$ has nowhere vanishing sections, whereas it may happen that every section of $E \boxtimes E^*$ vanishes at some point (x, x) of the diagonal. This explains some complications in the description of the kernels of Toeplitz operators.

6.2. The algebra $\mathcal{F}_r$

Recall that (M_r, ω_r) is a compact Kähler reduced orbifold with a prequantum bundle $L_r^\kappa \to M_r$ whose curvature is $-i\kappa\omega_r$. Let us consider a family $(\mathrm{T}_k)_k$ of operators, with Schwartz kernels

$$T_k \in C^\infty\big(M_r^2, L_r^k \boxtimes L_r^{-k}\big), \quad k = \kappa, 2\kappa, 3\kappa, \ldots.$$

As in the manifold case (cf. Section 5.2), the definition of the operators of $\mathcal{F}_r$ consists in two parts. The first assumption is

Assumption 6.1. T_k is $O_\infty(k^{-\infty})$ on every compact set $K \subset M_r^2$ such that $K \cap \Delta_r = \emptyset$.

The description of T_k on a neighborhood of the diagonal does not generalize directly from the manifold case, because the definition of the section $E^\kappa_{\Delta_r}$ does not make sense. Fortunately we can keep the same ansatz on the orbifold charts. If $(|U|, U, G, \pi_U)$ is an orbifold chart of M_r, we assume that:

Assumption 6.2. There exists a section $T'_{k,U}$ of $L^k_{r,U} \boxtimes L^{-k}_{r,U}$ invariant with respect to the diagonal action of G and of the form

$$T'_{k,U}(x, y) = \left(\frac{k}{2\pi}\right)^{n_r} E^k_{\Delta_U}(x, y) f(x, y, k) + O_\infty\big(k^{-\infty}\big) \tag{30}$$

on a neighborhood of the diagonal Δ_U, where $E^\kappa_{\Delta_U}$ and $f(\cdot, k)$ satisfy the assumptions (26)(i) and (26)(ii) with $(\Gamma, t^\kappa_\Gamma) = (\Delta_U, t^\kappa_{\Delta_U})$, such that

$$T_{k,UU} = \sum_{g \in G} (\mathcal{A}_g \boxtimes \mathrm{Id})^* T'_{k,U}. \tag{31}$$

Observe that we can use $T'_{k,U}$ instead of $T_{k,UU}$ to compute $(\mathrm{T}_k\Psi)_U$ when Ψ has compact support in U:

$$(\mathrm{T}_k\Psi)_U(x) = \frac{1}{\#G}\int_U T_{k,UU}(x,y)\Psi_U(y)\delta_U(y) = \int_U T'_{k,U}(x,y)\Psi_U(y)\delta_U(y). \tag{32}$$

This follows from (31) and the fact that Ψ_U is G-invariant and $\mathrm{T}'_{k,U}$ is invariant with respect to the diagonal action.

Definition 6.3. $\mathcal{F}_r$ is the set of operators (T_k) such that

$$\Pi_{r,k}\mathrm{T}_k\Pi_{r,k} = \mathrm{T}_k$$

and whose Schwartz kernel satisfies Assumptions 6.1 and 6.2 for every orbifold chart $(|U|, U, G, \pi_U)$.

The basic result is the generalization of the theorem of Boutet de Monvel and Sjöstrand on the Szegö projector.

Theorem 6.4. *The projector Π_k is an operator of $\mathcal{F}_r$.*

This theorem will be proved in Section 7. Let us deduce from it the properties of $\mathcal{F}_r$. First we define the total symbol map

$$\sigma : \mathcal{F}_r \to C^\infty(M_r)[\![\hbar]\!],$$

which sends T_k into the formal series $\sum \hbar^l g_l$ such that

$$T_k(x,x) = \left(\frac{k}{2\pi}\right)^{n_r} \sum k^{-l} g_l(x) + O\big(k^{-\infty}\big) \tag{33}$$

for every x which belongs to the principal stratum of M_r.

Proposition 6.5. *σ is well defined and onto. Furthermore, $\sigma(\mathrm{T}_k) = 0$ if and only if T_k is smoothing in the sense of Section* 5.1.1*, i.e. its kernel is $O_\infty(k^{-\infty})$.*

Proof. First let us prove that σ is well defined. By (31),

$$T_{k,UU}(x,x) = \sum_{g\in G} (\mathcal{A}_g \otimes \mathrm{Id})^* T'_{k,U}(a_g.x, x).$$

Assume that $\pi_U(x)$ belongs to the principal stratum of M_r. So if $g \neq \mathrm{id}_G$, $a_g.x \neq x$ which implies that $T'_k(a_g.x, x) = O(k^{-\infty})$. Consequently (30) leads to

$$T_{k,UU}(x,x) = \left(\frac{k}{2\pi}\right)^{n_r} f(x,x,k) + O\big(k^{-\infty}\big).$$

This proves the existence of the asymptotic expansion (33) and that the g_l extend to C^∞ functions on M_r. Furthermore, since the principal stratum is dense in M_r, these functions are uniquely determined by the kernel T_k. If T_k is smoothing, they vanish. Conversely, if the functions g_l vanish, $T'_{k,U}$ is $O_\infty(k^{-\infty})$ and the same holds for $T_{k,UU}$. So the kernel of σ consists of the T_k such that T_k is $O_\infty(k^{-\infty})$.

Let us prove that σ is onto. Let $\sum \hbar^l g_l$ be a formal series of $C^\infty(M_r)[\![\hbar]\!]$. First we construct a Schwartz kernel T_k satisfying Assumptions 6.1, 6.2 and (33). To do this, we introduce on every orbifold chart a kernel $T'_{k,U}$ of the form (30) where the functions $f(\cdot,k)$ are such that

$$f(\cdot,k) = \sum k^{-l} g_{l,U} + O\big(k^{-\infty}\big)$$

with the $g_{l,U} \in C^\infty(U)$ corresponding to the g_l. The existence of $T'_{k,U}$ is a consequence of the Borel lemma as in the manifold case. Furthermore, since the functions $g_{l,U}$ are G-invariant and the diagonal action of G preserves Δ_U and $t^\kappa_{\Delta_U}$, we can obtain a G-invariant section $T'_{k,U}$.

Then we piece together the $T_{k,UU}$ by using a partition of unity subordinate to a cover of M_r by orbifold charts. To do this, we have to check that two kernels T_{k,U_1U_1} and T_{k,U_2U_2} on two orbifold charts

$$\big(|U_1|, U_1, G_1, \pi_{U_1}\big), \qquad \big(|U_2|, U_2, G_2, \pi_{U_2}\big)$$

define the same section over $|U_1| \cap |U_2|$ modulo $O_\infty(k^{-\infty})$. Recall that the compatibility between orbifold charts is expressed by using orbifold charts $(|U|, U, G, \pi_U)$ which embed into $(|U_i|, U_i, G_i, \pi_{U_i})$. Denote by $\rho_i : G \to G_i$ the injective group homomorphism and by $j_i : U \to U_i$ the ρ_i-equivariant embeddings. We have

$$\forall g \in G_i, \forall x \in j_i(U), \quad a_g.x \in j_i(U) \quad \Rightarrow \quad g \in \rho_i(G). \tag{34}$$

We have to prove that

$$(j_i \boxtimes j_i)^* T_{k,U_iU_i} = T_{k,UU} + O_\infty\big(k^{-\infty}\big). \tag{35}$$

By (34), if $g \in G_i - \rho_i(G)$, then $a_g(j_i(U)) \cap j_i(U) = \emptyset$. Since T'_{k,U_i} is $O_\infty(k^{-\infty})$ outside the diagonal, this implies

$$T_{k,U_iU_i}|_{j_i(U)\times j_i(U)} = \sum_{g\in\rho_i(G)} (\mathcal{A}_g \boxtimes \mathrm{Id})^* T'_{k,U_i}|_{j_i(U)\times j_i(U)} + O_\infty\big(k^{-\infty}\big).$$

Furthermore, since $j_i^* g_{l,U_i} = g_{l,U}$, we have

$$(j_i \boxtimes j_i)^* T'_{k,U_i} = T'_{k,U} + O_\infty\big(k^{-\infty}\big).$$

Both of the previous equation lead to (35) by using that the map j_i are ρ_i-equivariant.

The section T_k obtained is the Schwartz kernel of an operator T_k. We do not have necessarily $\Pi_k \mathrm{T}_k \Pi_k = \mathrm{T}_k$, but only

$$\Pi_k \mathrm{T}_k \Pi_k \equiv \mathrm{T}_k \tag{36}$$

modulo an operator whose kernel is $O_\infty(k^{-\infty})$. So we replace T_k with $\Pi_k \mathrm{T}_k \Pi_k$. The proof of (36) is a consequence of Theorem 6.4 and is similar to the manifold case. Actually, if two operators R_k and S_k satisfy Assumptions 6.1 and 6.2, the kernel T_k of their product is given on a orbifold chart by

$$T_{k,UU} = \sum_{g \in G} (\mathcal{A}_g \boxtimes \mathrm{Id})^* T'_{k,U},$$

where

$$T'_{k,U}(x,y) = \int_U S'_{k,U}(x,z).R'_{k,U}(z,y)\, \delta_U(z) + O_\infty\big(k^{-\infty}\big). \tag{37}$$

This follows from Eq. (32). □

Theorem 6.6. *$\mathcal{F}_r$ is a $*$-algebra and the induced product of total symbols is a star-product of $C^\infty(M_r)[\![\hbar]\!]$. Every operator of $\mathcal{F}_r$ is of the form*:

$$\Pi_{r,k} M_{f(\cdot,k)} \Pi_{r,k} + O\big(k^{-\infty}\big), \tag{38}$$

where $f(\cdot,k)$ is a symbol of $S(M_r)$ and conversely. The map

$$E : C^\infty(M_r)[\![\hbar]\!] \to C^\infty(M_r)[\![\hbar]\!]$$

which sends the formal series $\sum \hbar^l f_l$ corresponding to the multiplicator $f(\cdot,k)$ into the total symbol of the operator (38)*, is well defined. It is an equivalence of star-products.*

Proof. As we have seen in the previous proof, the composition of kernels of operators of $\mathcal{F}_r$ corresponds in a orbifold chart to a composition on a manifold (cf. Eq. (37)). So the proof in the manifold case [6] extends directly. Let us recall the main steps. We first prove that $\mathcal{F}_r$ is a $*$-algebra and compute the product of the total symbols by applying the stationary phase lemma to the composition of the kernels. In the same way, we can compute the kernel of the operator (38), since $\Pi_{r,k}$ belongs to $\mathcal{F}_r$ by Theorem 6.4. As a result of the computation, this operator belongs to $\mathcal{F}_r$ and we obtain that the map E is an equivalence of star-product. From this we deduce that conversely every operator of $\mathcal{F}_r$ is of the form (38). □

6.3. Spectral density of a Toeplitz operator

Let T_k be a self-adjoint Toeplitz operator over M_r. Denote by d_k the dimension of $\mathcal{H}_{r,k}$ and by $E_1 \leqslant E_2 \leqslant \cdots \leqslant E_{d_k}$ the eigenvalues of T_k. The spectral density of T_k is the measure of $\mathbb{R}$

$$\mu_{\mathrm{T}_k}(E) = \sum_{i=1}^{d_k} \delta(E - E_i).$$

Let f be a C^∞ function on $\mathbb{R}$. We will estimate $\langle \mu_{\mathrm{T}_k}, f\rangle$. The first step is to compute the operator $f(\mathrm{T}_k)$.

Theorem 6.7. *If f is a C^∞ function on $\mathbb{R}$, then $f(\mathrm{T}_k)$ is a Toeplitz operator. Furthermore, if g_0 is the principal symbol of T_k then $f(g_0)$ is the principal symbol of $f(\mathrm{T}_k)$.*

The proof is similar to the manifold case (cf. [6, Proposition 12]). Now we have

$$\langle \mu_{\mathrm{T}_k}, f\rangle = \sum_{i=1}^{d_k} f(E_i) = \operatorname{Tr} f(\mathrm{T}_k).$$

By the previous theorem, it suffices to estimate the trace of a Toeplitz operator. Recall that

$$\operatorname{Tr}\mathrm{T}_k = \int_{M_r} T_k(x,x)\delta_{M_r}(x).$$

Let us begin with the computation in a orbifold chart $(|U|, U, G, \pi_U)$. Let η be a C^∞ function of M_r whose support is included in $|U|$. It follows from Assumption 6.2 that

$$\int_{M_r} \eta(x) T_k(x,x)\delta_{M_r}(x) = \frac{1}{\#G}\sum_{g\in G} I(g,U),$$

where

$$I(g,U) = \int_U \eta_U(x)(\mathcal{A}_g \boxtimes \mathrm{Id})^* T'_{k,U}(x,x)\delta_U(x).$$

Let U^g be the fixed point set of g,

$$U^g = \{x\in U;\ a_g.x = x\}. \tag{39}$$

Assume that U^g is connected. Then U^g is Kähler submanifold of U. Denote by $d(g)$ its complex codimension.

Let $y\in U^g$. $a_g : U\to U$ induces a linear transformation on the normal space $N_y = T_yU/T_yU^g$. It is a unitary map with eigenvalues

$$b_1(g), \dots, b_{d(g)}(g)$$

on the unit circle and not equal to 1. Furthermore, $\mathcal{A}_g$ acts on the fiber of $L^\kappa_{r,U}$ at y by multiplication by $c^\kappa(g)$, where $c(g)$ is on the unit circle. Since U^g is connected, the complex numbers $b_i(g)$ and $c(g)$ do not depend on y.

Lemma 6.8. *The integral $I(g,U)$ admits an asymptotic expansion*

$$I(g,U) = \left(\frac{k}{2\pi}\right)^{n_r - d(g)} c(g)^{-k}\sum_{l=0}^{\infty} k^{-l} I_l(g,U) + O\big(k^{-\infty}\big).$$

The first coefficient is given by

$$I_0(g,U) = \left(\prod_{i=1}^{d(g)} \big(1 - b_i(g)\big)^{-1}\right) \int_{U^g} \eta_U(x) f_{0,U}(x) \delta_{U^g}(x),$$

where f_0 is the principal symbol of T_k and δ_{U^g} is the Liouville measure of U^g.

Proof. Since $|T'_k(x,y)|$ is $O(k^{-\infty})$ if $x \neq y$, we can restrict the integral over a neighborhood of U^g. Let $y \in U^g$. Let (w^i) be a system of complex coordinates of U centered at y. Since g is of finite order, we can linearize the action of a_g. So we can assume that

$$a_g : (w_i)_{i=1,\dots,n_r} \to (w_1 b_1, \dots, w_d b_d, w_{d+1}, \dots, w_{n_r}).$$

To simplify the notations we denoted by d, b_i the numbers $d(g)$, $b_i(g)$. In the same way, we can choose a local holomorphic section s_r^κ of $L^\kappa_{r,U}$, which does not vanish on a neighborhood of y and such that

$$\mathcal{A}_g^* s_r^\kappa = c^{-\kappa}(g) s_r^\kappa.$$

Let H_r be the function such that $(s_r^\kappa, s_r^\kappa) = e^{-\kappa H_r}$. Observe that $H_r(a_g.x) = H_r(x)$. Denote by t^κ the unitary section $e^{\kappa H/2} s^\kappa$. Introduce a function $\tilde{H}_r(x,y)$ defined on U^2 such that $\tilde{H}_r(x,x) = H_r(x)$ and

$$(0, \bar{Z}).\tilde{H}_r \equiv 0 \quad \text{and} \quad (Z, 0).\tilde{H}_r \equiv 0$$

modulo $O(|x-y|^\infty)$ for every holomorphic vector field Z of U. Using that

$$\nabla t^\kappa = \frac{\kappa}{2}(\bar\partial H - \partial H) \otimes t^\kappa$$

we compute the section $E^k_{\Delta_U}$ of (30) (cf. [6, Proposition 1]), and obtain

$$T'_k(x,y) = \left(\frac{k}{2\pi}\right)^{n_r} e^{-k(\frac{1}{2}(H_r(x)+H_r(y)) - \tilde{H}_r(x,y))} f(x,y,k) t_r^k(x) \otimes t_r^{-k}(y).$$

Consequently the integral $I(g,U)$ is equal to

$$\left(\frac{k}{2\pi}\right)^{n_r} c^{-k}(g) \int_U e^{-k\phi_g(x)} f(a_g.x, x, k) \delta_U(x),$$

where $\phi_g(x) = H_r(x) - \tilde{H}_r(a_g.x, x)$. We estimate it by applying the stationary phase lemma. ϕ_g vanishes along U^g. Using that $a_g^* H_r = H_r$, we obtain for $i, j = 1, \dots, d$,

$$\partial_{w_i}\phi_g = \partial_{\bar{w}_i}\phi_g = \partial_{w_i}\partial_{w_j}\phi_g = \partial_{\bar{w}_i}\partial_{\bar{w}_j}\phi_g = 0$$

along U^g for $i, j = 1, \dots, d$. Denote by $H_{i,j}$ the second derivative $\partial_{w_i}\partial_{\bar{w}_j} H_r$. Then

$$\partial_{w_i}\partial_{\bar{w}_j}\phi_g = (1 - b_i) H_{i,j}$$

along U^g for $i, j = 1, \dots, d$. Furthermore,

$$\delta_U(x) = \det[H_{i,j}]_{i,j=1,\dots,n_r} |dw_1 d\bar{w}_1 \dots dw_{n_r} d\bar{w}_{n_r}|$$

which leads to the result. □

The next step is to patch together these local contributions. This involves a family of orbifolds associated to M_r which appears also in the Riemann–Roch theorem or in the definition of orbifold cohomology groups (cf. the associated orbifold of [19], the twisted sectors of [8], the inertia orbifold of [20]). The description of these orbifolds in the general case is rather complicated. Here, M_r is the quotient of P by a torus action, which simplifies the exposition (cf. [15, the appendix]).

Consider the following set:

$$\tilde{P} = \{(x, g) \in P \times \mathbb{T}^d;\ l_g.x = x\}.$$

To each connected component C of $\tilde{P}$ is associated a element g of $\mathbb{T}^d$ and a *support* $\bar{C} \subset P$ such that $C = \bar{C} \times \{g\}$. $\bar{C}$ is a closed submanifold of P invariant with respect to the action of $\mathbb{T}^d$. The quotient

$$F := \bar{C}/\mathbb{T}^d$$

is a compact orbifold, which embeds into M_r. Since $\mathbb{T}^d$ does not necessarily act effectively on $\bar{C}$, F is not in general a reduced orbifold. Denote by $m(F)$ its multiplicity.

Let $(x, g) \in C$ and denote by G the isotropy group of x. Let $U \subset P$ be a slice at x of the $\mathbb{T}^d$-action. Let $|U| = p(U)$ and π_U be the projection $U \to |U|$. Then $(|U|, U, G, \pi_U)$ is an orbifold chart of M_r. Introduce as in (39) the subset U^g of U and assume it is connected. Then $U^g = U \cap \bar{C}$. Let $|U^g| = p(U^g)$ and π_{U^g} be the projection $U^g \to |U^g|$. Then $(|U^g|, U^g, G, \pi_{U^g})$ is an orbifold chart of F. So $I_0(g, U)$ in Lemma 6.8 is given by an integral over F. Furthermore, since F is connected, there exists complex numbers

$$b_1(g, F), \dots, b_{d(F)}(g, F) \quad \text{and} \quad c(g, F)$$

on the unit circle corresponding to the numbers defined locally, with $d(F)$ the codimension of F in M_r. Observe also that F inherits a Kähler structure.

Denote by $\mathcal{F}$ the set

$$\mathcal{F} := \{\bar{C}/\mathbb{T}^d; \bar{C} \times \{g\} \text{ is a component of } \tilde{P}\}.$$

For every $F \in \mathcal{F}$, let $\mathbb{T}^d_F$ be the set of $g \in \mathbb{T}^d$ such that $F = \bar{C}/\mathbb{T}^d$ and $\bar{C} \times \{g\}$ is a component of $\tilde{P}$. The point is that two components of $\tilde{P}$ may have the same support. Since the set of components of $\tilde{P}$ is finite, the various sets $\mathcal{F}$ and $\mathbb{T}^d_F$ are finite.

Theorem 6.9. *Let* T_k *be a self-adjoint Toeplitz operator on* M_r *with principal symbol* g_0. *Let* f *be a* C^∞ *function on* $\mathbb{R}$. *Then* $\langle \mu_{\mathrm{T}_k}, f\rangle$ *admits an asymptotic expansion of the form*

$$\sum_{F\in\mathcal{F}} \left(\frac{k}{2\pi}\right)^{n_r-d(F)} \sum_{g\in\mathbb{T}^d_F} c(g,F)^{-k} \sum_{l=0}^{\infty} k^{-l} I_l(F,g) + O\left(k^{-\infty}\right),$$

where the coefficients $I_l(F,g)$ *are complex numbers. Furthermore,*

$$I_0(F,g) = \frac{1}{m(F)}\left(\prod_{i=1}^{d(F)}\left(1-b_i(g,F)\right)^{-1}\right)\int_F f(g_0)\delta_F,$$

where δ_F *is the Liouville measure of* F.

Remark 6.10. Each orbifold $F\in\mathcal{F}$ is the closure of a strata of M_r (cf. [15, the appendix]). For instance, M_r itself belongs to $\mathcal{F}$ and is the closure of the principal stratum of M_r. Note that $\mathbb{T}^d_{M_r}=\{0\}$ and that the other suborbifolds of $\mathcal{F}$ have a positive codimension. So at first order,

$$\langle \mu_{\mathrm{T}_k}, f\rangle = \left(\frac{k}{2\pi}\right)^{n_r}\int_{M_r} f(g_0)\delta_{M_r} + O\left(k^{n_r-1}\right).$$

Remark 6.11. Riemann–Roch–Kawasaki theorem gives the dimension of $\mathcal{H}_{r,k}$ in terms of characteristic forms, when k is sufficiently large:

$$\dim\mathcal{H}_{r,k} = \sum_{F\in\mathcal{F}}\sum_{g\in\mathbb{T}^d_F}\frac{1}{m(F)}\int_F \frac{\mathrm{Td}(F)\,\mathrm{Ch}(L_r^k,F,g)}{\mathrm{D}(N_F,g)}.$$

For the definition of these forms, we refer to [19, Theorem 3.3]. Let us compare this with the estimate of $\mathrm{Tr}(\Pi_k)$ given by Theorem 6.9. First, $\mathrm{Ch}(L_r^k,F,g)\in\Omega(F)$ is a twisted characteristic form associated to the pull-back of L_r^k by the embedding $F\to M_r$. At first order

$$\mathrm{Ch}\left(L_r^k,F,g\right) = c(g,F)^k\left(\frac{k}{2\pi}\right)^{n_F}\frac{\omega_F^{\wedge n_F}}{n_F!} + O\left(k^{n_F-1}\right),$$

where $n_F=n_r-d(F)$ and ω_F is the symplectic form of F. $\mathrm{D}(N_F,g)$ is a twisted characteristic form associated to the normal bundle N_F of the embedding $F\to M_r$,

$$\mathrm{D}(N_F,g)\equiv\prod_{i=1}^{d(F)}\left(1-b_i(g,F)\right) \mod \Omega^{\bullet\geqslant 2}(F).$$

$\mathrm{Td}(F)\equiv 1$ modulo $\Omega^{\bullet\geqslant 2}(F)$ is the Todd form of F. Hence we recover the leading term $I_0(F,g)$ in Theorem 6.9.

Remark 6.12. (*Harmonic oscillator*) We can describe explicitly the various term of Theorem 6.9 and deduce Theorem 2.3. Denote by $\mathcal{P}$ the set of greatest common divisors of the families $(\mathfrak{p}_i)_{i\in I}$ where I runs over the subsets of $\{1,\dots,n\}$. For every $\mathfrak{p}\in\mathcal{P}$, define the subset $I(\mathfrak{p})$ of $\{1,\dots,n\}$

$$I(\mathfrak{p}) := \{i;\ \mathfrak{p} \text{ divides } \mathfrak{p}_i\}$$

and the symplectic subspace $\mathbb{C}_{\mathfrak{p}}$ of $\mathbb{C}^n$

$$\mathbb{C}_{\mathfrak{p}} := \bigl\{z\in\mathbb{C}^n;\ z_i = 0 \text{ if } i\notin I(\mathfrak{p})\bigr\}.$$

Denote by $M_{\mathfrak{p}}$ the quotient of $P\cap\mathbb{C}_{\mathfrak{p}}$ by the S^1-action. Then one can check the following facts: $\mathcal{F}=\{M_{\mathfrak{p}};\ \mathfrak{p}\in\mathcal{P}\}$, the multiplicity of $M_{\mathfrak{p}}$ is $\mathfrak{p}$, its complex dimension is $\#I(\mathfrak{p})-1$. Furthermore, we have

$$S^1_{M_{\mathfrak{p}}} = \bigl\{\zeta\in\mathbb{C}^*;\ \zeta^{\mathfrak{p}}=1 \text{ and } \forall i\notin I(\mathfrak{p}),\ \zeta^{\mathfrak{p}_i}\neq 1\bigr\},$$

Note that the $S^1_{M_{\mathfrak{p}}}$ are mutually disjoint. Theorem 2.3 follows by using that

$$G = \bigcup_{\mathfrak{p}\in\mathcal{P}} S^1_{M_{\mathfrak{p}}}$$

and if $\zeta\in S^1_{M_{\mathfrak{p}}}$, then $c(\zeta, M_{\mathfrak{p}})=\zeta$ and the $b_i(\zeta, M_{\mathfrak{p}})$ are the $\zeta^{\mathfrak{p}_i}$ with $i\notin I(\mathfrak{p})$.

For example, if $(\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3)=(2,4,3)$, then $\mathcal{P}=\{1,2,4,3\}$. There are four supports: $M_1=M_r$, M_2 which is 1-dimensional and M_4, M_3 which consist of one point. The subsets of S^1 associated to these supports are

$$S^1_1=\{1\},\qquad S^1_2=\{-1\},\qquad S^1_4=\{i,-i\}\quad\text{and}\quad S^1_3=\bigl\{e^{i\pi/3}, e^{i2\pi/3}\bigr\}.$$

7. The kernel of the Szegö projector of M_r

In this section, we prove that the Szegö projector $\Pi_{r,k}$ of the orbifold M_r is a Fourier integral operator, which is the content of Theorem 6.4. In the manifold case this result is a consequence of a theorem of Boutet de Monvel and Sjöstrand on the kernel of the Szegö projector associated to the boundary of a strictly pseudoconvex domain [5]. We will not adapt the proof of Boutet de Monvel and Sjöstrand. Instead we will deduce this result from the same result for the Szegö projector of M, i.e. we will show that

$$\Pi_k\in\mathcal{F}\quad\Rightarrow\quad\Pi_{r,k}\in\mathcal{F}_r.$$

First we define the algebra $\mathcal{F}_\lambda$, that we introduced in Section 5 for a free torus action. Then following an idea of Guillemin and Sternberg (cf. [13, the appendix]), we prove that $\Pi_{\lambda,k}\in\mathcal{F}_\lambda$. Then we relate the algebras $\mathcal{F}_\lambda$ and $\mathcal{F}_r$ as in Theorem 5.10 and deduce Theorem 6.4. To compare with Section 5, the proof of Theorem 5.5 was in reverse order, i.e. we showed that $\Pi_{r,k}\in\mathcal{F}_r\Rightarrow\Pi_{\lambda,k}\in\mathcal{F}_\lambda$.

We think that the ansatz we propose for the kernel of $\Pi_{r,k}$ is also valid for a general Kähler orbifold, not obtained by reduction. But here, it is easier and more natural to deduce this by reduction.

7.1. The algebra $\mathcal{F}_\lambda$

Since the $\mathbb{T}^d$-action is not necessarily free, we have to modify the definition of the algebra $\mathcal{F}_\lambda$. We consider M as an orbifold and give local ansatz in orbifold charts as we did with the Toeplitz operators in Section 6.2.

Introduce as in Remark 3.8 an orbifold chart $(|U|, U, G, \pi_U)$ of M_r with the associated orbifold chart $(|V|, V, G, \pi_V)$ of $P_{\mathbb{C}}$. Denote by a_g (respectively l_θ) the action of $g \in G$ (respectively $\theta \in \mathbb{T}^d$) on V. These actions both lift to $L_V^\kappa := \pi_V^* L^\kappa$. We denote them by $\mathcal{A}_g$ and $\mathcal{L}_\theta$.

Define the data $(\Lambda_V, t^\kappa_{\Lambda_V})$ corresponding to the data $(\Lambda, t^\kappa_\Lambda)$ of Section 5.3.

$$\Lambda_V := \{(\theta, 0, u, \theta', 0, u) \in V^2;\ \theta, \theta' \in \mathbb{T}^d \text{ and } u \in U\},$$
$$t^\kappa_{\Lambda_V}(\theta, 0, u, \theta', 0, u) := \mathcal{L}_{\theta-\theta'}.z \otimes z^{-1} \quad \text{if } z \in L^\kappa_{V,(\theta',0,u)} \text{ and } z \neq 0.$$

Note that $t^\kappa_{\Lambda_V}$ is well defined because the $\mathbb{T}^d$-action on V is free. In the definition of an operator T_k of $\mathcal{F}_\lambda$, we will assume that the lift $T_{k,VV}$ of its Schwartz kernel T_k to V^2 satisfies

Assumption 7.1. There exists a section $T'_{k,V}$ of $L_V^k \boxtimes L_V^{-k}$ invariant with respect to the action of $\mathbb{T}^d \times \mathbb{T}^d$ and the diagonal action of G and of the form

$$T'_{k,V}(x, y) = \left(\frac{k}{2\pi}\right)^{n-d/2} E^k_{\Lambda_V}(x, y) f(x, y, k) + O_\infty\big(k^{-\infty}\big) \tag{40}$$

on a neighborhood of Λ_V, where $E^\kappa_{\Lambda_V}$ and $f(\cdot, k)$ satisfy the assumptions (26)(i) and (26)(ii) with $(\Gamma, t^\kappa_\Gamma) = (\Lambda_V, t^\kappa_{\Lambda_V})$, such that

$$T_{k,VV} = \sum_{g\in G} (\mathcal{A}_g \boxtimes \mathrm{Id})^* T'_{k,V}.$$

The whole definition of an operator of $\mathcal{F}_\lambda$ is the following.

Definition 7.2. $\mathcal{F}_\lambda$ is the set of operators (T_k) with Schwartz kernel T_k such that:

- $\Pi_k \mathrm{T}_k \Pi_k = \mathrm{T}_k$ and $\mathcal{L}^*_\theta \mathrm{T}_k = \mathrm{T}_k \mathcal{L}^*_\theta = \mathrm{T}_k$, for all $\theta \in \mathbb{T}^d$;
- T_k is $O_\infty(k^{-\infty})$ on every compact set $K \subset M^2$ such that $K \cap \Lambda = \emptyset$;
- T_k satisfies Assumption 7.1 for every orbifold chart $(|V|, V, G, \pi_V)$.

By adapting the proof of Theorem 5.8, we define the symbol map.

Proposition 7.3. *There exists a map $\sigma : \mathcal{F}_\lambda \to C^\infty(M_r)[[\hbar]]$ which is onto and whose kernel consists of smoothing operators.*

The following theorem will be proved in the next subsection.

Theorem 7.4. *$\Pi_{\lambda,k}$ is an elliptic operator of $\mathcal{F}_\lambda$.*

Remark 7.5. $\Pi_{\lambda,k}$ is elliptic means that its principal symbol does not vanish. This implies that $\Pi_{\lambda,k}$ itself does not vanish when k is sufficiently large, so $\mathcal{H}_{\lambda,k}$ is not reduced to (0) when k is large enough. This completes the proof of the Guillemin–Sternberg Theorem 3.3 in the orbifold case.

In Proposition 6.5, we defined the total symbol map $\sigma : \mathcal{F}_r \to C^\infty(M_r)[\![\hbar]\!]$ and prove that its kernel consists of smoothing operators, without using that $\Pi_{r,k} \in \mathcal{F}_r$. We can now generalize Theorem 5.10.

Theorem 7.6. *The map $\mathcal{F}_\lambda \to \mathcal{F}_r$ which sends T_k into $\mathrm{V}_k \mathrm{T}_k \mathrm{V}_k^*$ is well defined and bijective. Furthermore, the total symbols of $\mathrm{T}_k \in \mathcal{F}_\lambda$ and $\mathrm{V}_k \mathrm{T}_k \mathrm{V}_k^*$ are the same.*

The proof follows the same line as the proof of Theorem 5.10. To every chart $(|V|, V, G, \pi_V)$ of M is associated a chart $(|U|, U, G, \pi_U)$. Assumption 7.1 corresponds to Assumption 6.2.

As a corollary of this theorem, the total symbol map $\sigma : \mathcal{F}_r \to C^\infty(M_r)[\![\hbar]\!]$ is onto. Furthermore, it follows from the stationary phase lemma that $\mathcal{F}_r$ is a $*$-algebra and the induced product $*_r$ on $C^\infty(M_r)[\![\hbar]\!]$ is a star-product.

Let $\tilde{\Pi}_{r,k}$ be an operator of $\mathcal{F}_r$ whose total symbol is the unit of $(C^\infty(M_r)[\![\hbar]\!], *_r)$.

Lemma 7.7. $(\mathrm{W}_k \tilde{\Pi}_{r,k} \mathrm{W}_k^*)(\mathrm{V}_k^* \mathrm{V}_k) = \Pi_{\lambda,k} + \mathrm{R}_k$, *where* R_k *is* $O(k^{-\infty})$.

Proof. Since $\mathrm{W}_k^* \mathrm{V}_k^* = \Pi_{r,k}$ and $\mathrm{V}_k^* \mathrm{W}_k^* = \Pi_{\lambda,k}$, we have

$$\mathrm{W}_k \tilde{\Pi}_{r,k} \mathrm{W}_k^* \mathrm{V}_k^* \mathrm{V}_k = \mathrm{W}_k \tilde{\Pi}_{r,k} \mathrm{V}_k = \mathrm{W}_k \tilde{\Pi}_{r,k} \mathrm{V}_k \mathrm{V}_k^* \mathrm{W}_k^*.$$

By Theorem 7.6, $\mathrm{V}_k \mathrm{V}_k^* = \mathrm{V}_k \Pi_{\lambda,k} \mathrm{V}_k^*$ belongs to $\mathcal{F}_r$ since $\Pi_{\lambda,k}$ belongs to $\mathcal{F}_\lambda$. From the symbolic calculus of the operators of $\mathcal{F}_r$, it follows that

$$\tilde{\Pi}_{r,k}\big(\mathrm{V}_k \mathrm{V}_k^*\big) \equiv \big(\mathrm{V}_k \mathrm{V}_k^*\big)$$

modulo an operator of $\mathcal{F}_r$ whose total symbol vanishes. Applying again Theorem 7.6, we obtain that

$$\mathrm{W}_k\big(\tilde{\Pi}_{r,k}(\mathrm{V}_k \mathrm{V}_k^*)\big)\mathrm{W}_k^* \equiv \mathrm{W}_k \mathrm{V}_k \mathrm{V}_k^* \mathrm{W}_k^*$$

modulo an operator of $\mathcal{F}_\lambda$ whose total symbol vanishes. The left-hand side is equal to $\mathrm{W}_k \tilde{\Pi}_{r,k} \mathrm{W}_k^* \mathrm{V}_k^* \mathrm{V}_k$ and the right-hand side to $\Pi_{\lambda,k}$. This proves Lemma 7.7. □

Denote by $(\mathrm{V}_k^* \mathrm{V}_k)^{-1}$ the inverse of $\mathrm{V}_k^* \mathrm{V}_k$ on $\mathcal{H}_{\lambda,k}$, that is

$$\Pi_{\lambda,k}\big(\mathrm{V}_k^* \mathrm{V}_k\big)^{-1} \Pi_{\lambda,k} = \big(\mathrm{V}_k^* \mathrm{V}_k\big)^{-1} \quad \text{and}$$
$$\big(\mathrm{V}_k^* \mathrm{V}_k\big)\big(\mathrm{V}_k^* \mathrm{V}_k\big)^{-1} = \big(\mathrm{V}_k^* \mathrm{V}_k\big)^{-1}\big(\mathrm{V}_k^* \mathrm{V}_k\big) = \Pi_{\lambda,k}.$$

Lemma 7.7 implies that

$$\big(\mathrm{V}_k^* \mathrm{V}_k\big)^{-1} = \mathrm{W}_k \tilde{\Pi}_{r,k} \mathrm{W}_k^* - \mathrm{R}_k \big(\mathrm{V}_k^* \mathrm{V}_k\big)^{-1}. \tag{41}$$

By Theorem 7.6, $\mathrm{W}_k\tilde{\Pi}_{r,k}\mathrm{W}_k^*$ belongs to $\mathcal{F}_\lambda$.

Lemma 7.8. $\mathrm{R}_k(\mathrm{V}_k^*\mathrm{V}_k)^{-1}$ *is* $O(k^{-\infty})$.

Proof. Since $\mathrm{W}_k\tilde{\Pi}_{r,k}\mathrm{W}_k^*$ belongs to $\mathcal{F}_\lambda$, its kernel is $O(k^{n-d/2})$. So $\mathrm{W}_k\tilde{\Pi}_{r,k}\mathrm{W}_k^*$ is $O(k^{n-d/2})$. Using that R_k is $O(k^{-\infty})$, it follows from (41) that $(\mathrm{V}_k^*\mathrm{V}_k)^{-1}$ is $O(k^{n-d/2})$. □

We deduce from this that $(\mathrm{V}_k^*\mathrm{V}_k)^{-1} \in \mathcal{F}_\lambda$. We have

$$\Pi_{r,k} = \mathrm{V}_k\big(\mathrm{V}_k^*\mathrm{V}_k\big)^{-1}\mathrm{V}_k^*.$$

Consequently Theorem 7.6 implies that $\Pi_{r,k} \in \mathcal{F}_r$.

7.2. The projector $\Pi_{\lambda,k}$

This section is devoted to the proof of Theorem 7.4. We use that $\Pi_k \in \mathcal{F}$ together with the following consequence of (16):

$$\Pi_{\lambda,k}(\underline{x}, x) = \int_{\mathbb{T}^d} \big((\mathcal{L}_\theta \boxtimes \mathrm{Id})^*\Pi_k\big)(\underline{x}, x)\delta_{\mathbb{T}^d}(\theta). \tag{42}$$

The only difficult point is to prove that $\Pi_{\lambda,k}$ satisfies Assumption 7.1.

Introduce an orbifold chart $(|V|, V, G, \pi_V)$ of M as in the previous section. Denote by $\Pi_{\lambda,k,VV}$ and $\Pi_{k,VV}$ the lifts of $\Pi_{\lambda,k}$ and Π_k to V^2. Then

$$\Pi_{\lambda,k,VV}(\underline{v}, v) = \int_{\mathbb{T}^d} \big((\mathcal{L}_\theta \boxtimes \mathrm{Id})^*\Pi_{k,VV}\big)(\underline{v}, v)\delta_{\mathbb{T}^d}(\theta).$$

We know that $\Pi_k \in \mathcal{F}$. So $\Pi_{k,V}$ satisfies Assumption 6.2 for the orbifold M. Denote by $\Pi'_{k,V}$ an associated kernel such that

$$\Pi_{k,VV} = \sum_{g\in G}(\mathcal{A}_g \boxtimes \mathrm{Id})^*\Pi'_{k,V}.$$

If we prove that $\Pi'_{\lambda,k,V}$ given by

$$\Pi'_{\lambda,k,V}(\underline{v}, v) = \int_{\mathbb{T}^d} \big((\mathcal{L}_\theta \boxtimes \mathrm{Id})^*\Pi'_{k,V}\big)(\underline{v}, v)\delta_{\mathbb{T}^d}(\theta) \tag{43}$$

satisfies (40), we are done. So the proof is locally reduced to the manifold case.

Let us relate the section $E^\kappa_{\Delta_V}$ and $E^\kappa_{\Lambda_V}$ appearing in (30) and (40). Let

$$s^\kappa : V \to L^\kappa_V$$

be a holomorphic $\mathbb{T}^d_{\mathbb{C}}$-invariant section which does not vanish. Introduce the real function H such that $(s^\kappa, s^\kappa)(v) = e^{-\kappa H(v)}$ and the unitary section $t^\kappa = e^{\kappa H/2} s^\kappa$. Let us write

$$E^\kappa_{\Delta_V}(\underline{v}, v) = e^{-\kappa \phi_\Delta(\underline{v}, v)} t^\kappa(\underline{v}) \otimes t^\kappa(v), \qquad E^\kappa_{\Lambda_V}(\underline{v}, v) = e^{-\kappa \phi_\Lambda(\underline{v}, v)} t^\kappa(\underline{v}) \otimes t^\kappa(v).$$

So we have

$$\Pi'_{k,V}(\underline{v}, v) = \left(\frac{k}{2\pi}\right)^n e^{-k\phi_\Delta(\underline{v}, v)} f(\underline{v}, v, k) t^k(\underline{v}) \otimes t^k(v) + O_\infty\big(k^{-\infty}\big).$$

Assumption (26)(i) determines only the Taylor expansion of E^κ_Γ along Γ. Hence, the functions ϕ_Δ and ϕ_Λ are unique modulo a function which vanishes to any order along the associated Lagrangian manifold.

Recall that we introduced a function φ in Proposition 4.8. Let $\tilde{\varphi}(\underline{v}, v)$ be a function such that such that $\tilde{\varphi}(v, v) = \varphi(v)$ and

$$\underline{\bar{Z}}\tilde{\varphi} \equiv Z.\tilde{\varphi} \equiv 0$$

modulo $O(|\underline{v} - v|^\infty)$ for every holomorphic vector field Z of V. Here $\underline{\bar{Z}}\tilde{\varphi}$ (respectively $Z.\tilde{\varphi}$) denote the derivative of $\tilde{\varphi}$ with respect to the vector field $(\bar{Z}, 0)$ of V^2 (respectively $(0, Z)$). We use the same notation in the following.

Lemma 7.9. *We can choose the functions ϕ_Λ and ϕ_Δ in such a way that*

$$\phi_\Delta(\underline{v}, v) = \phi_\Lambda(\underline{v}, v) - \tilde{\varphi}(\underline{v}, v).$$

Proof. Recall that $V = \mathbb{T}^d \times \mathfrak{t}_d \times U \ni (\theta, t, u) = v$. By Proposition 4.8, we have

$$H(v) = H_r(u) + \varphi(t, u). \tag{44}$$

By reduction, U is endowed with a complex structure (cf. Section 3.2). Introduce a function $\tilde{H}_r(\underline{u}, u)$ such that $\tilde{H}_r(u, u) = H_r(u)$ and

$$\underline{\bar{Z}}\tilde{H}_r \equiv Z.\tilde{H}_r \equiv 0 \mod O\big(|\underline{u} - u|^\infty\big)$$

for every holomorphic vector field Z of U. Then

$$\phi_\Lambda(\underline{v}, v) := \frac{1}{2}\big(H(\underline{v}) + H(v)\big) - \tilde{H}_r(\underline{u}, u) \tag{45}$$

is a function associated to Λ_V. This is easily checked using that

$$\nabla t^\kappa = \frac{\kappa}{2}(\bar{\partial} H - \partial H) \otimes t^\kappa.$$

Set $\tilde{H}(\underline{v}, v) = \tilde{H}_r(\underline{u}, u) + \tilde{\varphi}(\underline{v}, v)$. In the same way we get that

$$\phi_\Delta(\underline{v}, v) := \frac{1}{2}\big(H(\underline{v}) + H(v)\big) - \tilde{H}(\underline{v}, v)$$

is a function associated to Δ_V. $\square$

Lemma 7.10. *We have over V^2*

$$\Pi'_{\lambda,k,V}(\underline{v}, v) = \left(\frac{k}{2\pi}\right)^{n-d/2} e^{-k\phi_\Lambda(\underline{v},v)} g(\underline{v}, v, k) t^k(\underline{v}) \otimes t^k(v) + O_\infty\big(k^{-\infty}\big),$$

where

$$g(\underline{v}, v, k) = \left(\frac{k}{2\pi}\right)^{d/2} \int_W e^{k\phi(\theta',\underline{v},v)} f(\theta+\theta', \underline{t}, \underline{u}, v, k)|d\theta'|,$$

W is any neighborhood of 0 *in* $\mathbb{T}^d$ *and* $\phi(\theta', \underline{v}, v) = \tilde{\varphi}(\theta+\theta', \underline{t}, \underline{u}, v)$.

Proof. Since $\mathcal{L}^*_\theta t^k = t^k$, (43) implies

$$\Pi'_{\lambda,k,V}(\underline{v}, v) \equiv \left(\frac{k}{2\pi}\right)^{n} t^k(\underline{v}) \otimes t^k(v) \int_{\mathbb{T}^d} e^{-k\phi_\Delta(\underline{\theta}+\theta',\underline{t},\underline{u},v)} f(\underline{\theta}+\theta', \underline{t}, \underline{u}, v, k)|d\theta'|$$

modulo $O_\infty(k^{-\infty})$. Replacing $\theta' + \underline{\theta} - \theta$ by θ', this leads to

$$\Pi'_{\lambda,k,V}(\underline{v}, v) \equiv \left(\frac{k}{2\pi}\right)^{n} t^k(\underline{v}) \otimes t^k(v) \int_{\mathbb{T}^d} e^{-k\phi_\Delta(\theta+\theta',\underline{t},\underline{u},v)} f(\theta+\theta', \underline{t}, \underline{u}, v, k)|d\theta'|$$

modulo $O_\infty(k^{-\infty})$. Since the imaginary part of Φ_Δ is positive outside the diagonal of V^2, we can restrict the integral over any neighborhood W of 0 in $\mathbb{T}^d$. Now using Lemma 7.9 and the fact that ϕ_Λ is independent of θ which appears in Eq. (45), we obtain the result. □

To complete the proof of Theorem 7.4, it suffices to prove that $g(\cdot, k)$ admits an asymptotic expansion in power of k for the C^∞ topology on a neighborhood $\Lambda \cap V^2$. We prove this by applying the stationary phase Lemma [16]. So the result is a consequence of the following lemma.

Lemma 7.11. *Let $(\underline{v}_0, v_0) \in \Lambda \cap V^2$. Then the Hessian $d^2_{\theta'}\phi$ at $(0, \underline{v}_0, v_0)$ is a real definite positive matrix. Furthermore, on a neighborhood of $(0, \underline{v}_0, v_0)$ in $\mathbb{T}^d \times V^2$,*

$$\phi = \sum h_{ij} (\partial_{\theta'^i}\phi)(\partial_{\theta'^j}\phi),$$

where the h_{ij} are C^∞ functions of θ', $\underline{v}$, v.

Before we prove this lemma, let us state some intermediate results. If h is a function of $C^\infty(V)$, we denote by $\tilde{h}$ a function of $C^\infty(V^2)$ such that $\tilde{h}(\underline{v}, v) = h(v)$ and

$$\bar{\underline{Z}}.\tilde{h} \equiv Z.\tilde{h} \equiv 0 \mod O\big(|\underline{v} - v|^\infty\big) \tag{46}$$

for every holomorphic vector field Z of V.

Denote by t^i the coordinates of $t = \sum t^i \xi_i \in \mathfrak{t}_d$. Let us compute the derivatives of $\tilde{h}$ with respect to the vector fields $\partial_{\theta j} = \xi^\#_j$ and $\partial_{tj} = J\xi^\#_j$ acting on the left and the right, respectively.

Lemma 7.12. *If h is $\mathbb{T}^d$-invariant, then*

$$i\partial_{\underline{\theta}^j}\tilde{h} \equiv \partial_{\underline{t}^j}\tilde{h} \equiv -i\partial_{\theta^j}\tilde{h} \equiv \partial_{t^j}\tilde{h} \equiv \frac{1}{2}\tilde{h}_j \mod O\big(|\underline{v}-v|^\infty\big),$$

where $h_j = \partial_{t^j}h$.

In particular, we obtain the following relations:

$$2\partial_{\underline{t}^j}\tilde{t}^i \equiv \delta_{ij}, \qquad 2\partial_{t^j}\tilde{t}^i \equiv \delta_{ij}, \qquad 2\partial_{\underline{\theta}^j}\tilde{t}^i \equiv -i\delta_{ij}, \qquad 2\partial_{\theta^j}\tilde{t}^i \equiv i\delta_{ij} \tag{47}$$

modulo $O(|\underline{v}-v|^\infty)$.

Proof. Since $[Z, \partial_{\theta^j}]$ (respectively $[Z, \partial_{t^j}]$) is a holomorphic vector field when Z is, the various derivatives we need to compute satisfy Eq. (46). So we just have to compute their restriction to the diagonal of V^2. To do this observe that

$$\langle d\tilde{h}, \partial_{\underline{\theta}^j} + i\partial_{\underline{t}^j}\rangle|_{(v,v)} = 0, \qquad \langle d\tilde{h}, \partial_{\theta^j} - i\partial_{t^j}\rangle|_{(v,v)} = 0$$

since $\partial_{\theta^j} - i\partial_{t^j} = \xi_j^\# - iJ\xi_j^\#$ is holomorphic. Furthermore,

$$\langle d\tilde{h}, \partial_{\underline{\theta}^j} + \partial_{\underline{\theta}^j}\rangle|_{(v,v)} = 0, \qquad \langle d\tilde{h}, \partial_{\underline{t}^j} + \partial_{t^j}\rangle|_{(v,v)} = \partial_{t^j}.h(v)$$

since $\tilde{h}(v,v) = h(v)$ and h is $\mathbb{T}^d$-invariant. □

Proof of Lemma 7.11. Let us write $\varphi(v) = \frac{1}{2}\sum t^i t^j \varphi_{ij}(v)$ on a neighborhood of v_0. Consequently, we have on a neighborhood of (v_0, v_0)

$$\tilde{\varphi} = \frac{1}{2}\sum \tilde{t}^i\tilde{t}^j\tilde{\varphi}_{ij}. \tag{48}$$

It follows from Eq. (47) and Proposition 4.10 that

$$\partial_{\underline{\theta}^i}\partial_{\underline{\theta}^j}\tilde{\varphi}(v_0,v_0) = -\varphi_{ij}(v_0) = -2g\big(\xi_i^\#, \xi_j^\#\big)(v_0).$$

The first part of the lemma follows. By (48) there exist C^∞ functions h_{ij}^1 such that $\partial_{\underline{\theta}^i}\tilde{\varphi} = \sum \tilde{t}^j h_{ij}^1$ on a neighborhood of (v_0, v_0). Derivating with respect to $\underline{\theta}$, we obtain

$$h_{ij}^1(v_0,v_0) = -i\varphi_{ij}(v_0).$$

Hence it is an invertible matrix and there exist functions h_{ij}^2 such that $\tilde{t}^i = \sum h_{ij}^2\partial_{\underline{\theta}^j}\tilde{\varphi}$ on a neighborhood of (v_0, v_0). By (48),

$$\tilde{\varphi} = \sum h_{ij}^3(\partial_{\underline{\theta}^i}\tilde{\varphi})(\partial_{\underline{\theta}^j}\tilde{\varphi})$$

for some C^∞ functions h_{ij}^3 on a neighborhood of (v_0, v_0). The second part of the lemma follows. □

Acknowledgments

I thank Sandro Graffi for his encouragement and Alejandro Uribe for introducing me to the quantization of symplectic reduction a few years ago.

References

[1] M.F. Atiyah, Convexity and commuting Hamiltonians, Bull. London Math. Soc. 14 (1) (1982) 1–15.
[2] F.A. Berezin, General concept of quantization, Comm. Math. Phys. 40 (1975) 153–174.
[3] D. Borthwick, T. Paul, A. Uribe, Semiclassical spectral estimates for Toeplitz operators, Ann. Inst. Fourier (Grenoble) 48 (4) (1998) 1189–1229.
[4] L. Boutet de Monvel, V. Guillemin, The Spectral Theory of Toeplitz Operators, Ann. of Math. Studies, vol. 99, Princeton Univ. Press, Princeton, NJ, 1981.
[5] L. Boutet de Monvel, J. Sjöstrand, Sur la singularité des noyaux de Bergman et de Szegő, in: Journées: Équations aux Dérivées Partielles de Rennes, 1975, Astérisque 34–35 (1976) 123–164.
[6] L. Charles, Berezin–Toeplitz operators, a semi-classical approach, Comm. Math. Phys. 239 (1–2) (2003) 1–28.
[7] L. Charles, Quasimodes and Bohr–Sommerfeld conditions for the Toeplitz operators, Comm. Partial Differential Equations 28 (9–10) (2003) 1527–1566.
[8] W. Chen, Y. Ruan, Orbifold Gromov–Witten theory, in: Orbifolds in Mathematics and Physics, Madison, WI, 2001, in: Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 25–85.
[9] J.J. Duistermaat, The momentum map, in: Topics in Differential Geometry, vols. I, II, Debrecen, 1984, in: Colloq. Math. Soc. János Bolyai, vol. 46, North-Holland, Amsterdam, 1988, pp. 347–392.
[10] J.J. Duistermaat, The Heat Kernel Lefschetz Fixed Point Formula for the Spin-c Dirac Operator, Birkhäuser Boston, Boston, MA, 1996.
[11] B. Fedosov, Deformation Quantization and Index Theory, Math. Topics, vol. 9, Akademie Verlag, Berlin, 1996.
[12] V. Guillemin, S. Sternberg, Convexity properties of the moment mapping, Invent. Math. 67 (3) (1982) 491–513.
[13] V. Guillemin, S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math. 67 (3) (1982) 515–538.
[14] V. Guillemin, S. Sternberg, Homogeneous quantization and multiplicities of group representations, J. Funct. Anal. 47 (3) (1982) 344–380.
[15] V. Guillemin, V. Ginzburg, Y. Karshon, Moment Maps, Cobordisms, and Hamiltonian Group Actions, Math. Surveys Monogr., vol. 98, Amer. Math. Soc., Providence, RI, 2002, Appendix J by M. Braverman.
[16] L. Hörmander, The Analysis of Linear Partial Differential Operators. I, Distribution Theory and Fourier Analysis, second edition, Springer-Verlag, Berlin, 1990.
[17] T. Kawasaki, The Riemann–Roch theorem for complex V-manifolds, Osaka J. Math. 16 (1) (1979) 151–159.
[18] J. Marsden, A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys. 5 (1) (1974) 121–130.
[19] E. Meinrenken, Symplectic surgery and the Spin^c-Dirac operator, Adv. Math. 134 (2) (1998) 240–277.
[20] I. Moerdijk, Orbifolds as groupoids: An introduction, in: Orbifolds in Mathematics and Physics, Madison, WI, 2001, in: Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 205–222.
[21] Y. Ruan, Stringy orbifolds, in: Orbifolds in Mathematics and Physics, Madison, WI, 2001, in: Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 259–299.
[22] I. Satake, On a generalization of the notion of manifold, Proc. Natl. Acad. Sci. USA 42 (1956) 359–363.
[23] R. Sjamaar, Symplectic reduction and Riemann–Roch formulas for multiplicities, Bull. Amer. Math. Soc. (N.S.) 33 (3) (1996) 327–338.
[24] W.P. Thurston, Three-Dimensional Geometry and Topology, vol. 1, S. Levy (Ed.), Princeton Math. Ser., vol. 35, Princeton Univ. Press, Princeton, NJ, 1997.
[25] A. Weinstein, Lectures on Symplectic Manifolds, in: CBMS Reg. Conf. Ser. Math., vol. 29, Amer. Math. Soc., Providence, RI, 1979, corrected reprint.
[26] P. Xu, Fedosov $*$-products and quantum momentum maps, Comm. Math. Phys. 197 (1) (1998) 167–197.